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Projective plane curves whose complements have $\overline{\kappa} = 1$

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Abstract

We consider an irreducible curve C with two cuspidal singular points on the projective plane $\mathbf{P}^2$ such that the complement $\mathbf{P}^2 - C$ has logarithmic Kodaira dimension one. Since $\mathbf{P}^2 - C$ is a $\mathbf{Q}$-homology plane, we have two cases to consider according to the unique reducible fiber of a $\mathbf{C}^*$-fibration on $\mathbf{P}^2 - C$. In the first case, the reducible fiber consists of two curves isomorphic to the affine line $\mathbf{A}^1$ and meeting each other in one point. In this case we can write down explicitly a defining equation of C. In the second case, the reducible fiber is a disjoint union of two curves, one of which is isomorphic to $\mathbf{C}^*$ and the other to $\mathbf{A}^1$. In the second case, we can give a defining equation under some minor additional hypotheses. The case where $\mathbf{P}^2 - C$ has logarithmic Kodaira dimension $-\infty$ was studied in [8].

0 Introduction

All algebraic varieties considered in this paper are defined over the field of complex numbers $\mathbf{C}$. Let C be an irreducible curve on the projective plane $\mathbf{P}^2$, which we simply call an irreducible plane curve. In order to analyze the curve C, it is important to consider logarithmic Kodaira dimension of its complement $X := \mathbf{P}^2 - C$, which we denote by $\overline{\kappa}(X)$ (see Iitaka [3] for the definition and the relevant results on logarithmic Kodaira dimension). Miyanishi and Sugie [8] considered an irreducible plane curve C with $\overline{\kappa}(\mathbf{P}^2 - C) = -\infty$ and determined possible types of such a curve by means of the theory of $\mathbf{A}^1$-rulings.

Meanwhile, it is known by Tsunoda [12] and Wakabayashi [13] that an irreducible plane curve C with $\deg C \geq 4$ has $\overline{\kappa}(\mathbf{P}^2 - C) = 2$ except for the following two cases:

(A) C is a rational curve with one singular point,

(B) C is a rational curve with two cuspidal singular points.

Tsunoda [12] showed that $\overline{\kappa}(\mathbf{P}^2 - C) = 1$ or 2 in the case (B) and that $\overline{\kappa}(\mathbf{P}^2 - C) \neq 0$ if C is a rational curve with only one cuspidal point.

In the present article we consider an irreducible plane curve C of $\overline{\kappa}(\mathbf{P}^2 - C) = 1$ and with two cuspidal points. To be specific, our problem is stated as follows:

(1) *Describe the structure of the complement* $X := \mathbf{P}^2 - C$ *via the existence of* $\mathbf{C}^*$*-fibrations, e.g., the number of singular fibers or multiple fibers and the distribution of multiplicities.*

(2) *Determine a homogeneous defining equation of* C *up to automorphisms of* $\mathbf{P}^2$ *by making use of the informations given in (1).*

If C is a rational plane curve with only cuspidal singular points, its complement X is a $\mathbf{Q}$*-homology plane*, which is by definition a smooth affine surface with $H_i(X;\mathbf{Q}) = 0$ for all $i > 0$. See Miyanishi and Sugie [9] for the relevant results on $\mathbf{Q}$-homology planes. If C is not rational or has singularities other than the cuspidal singularity, X is not a $\mathbf{Q}$-homology plane. Hence the above problem (2) can be stated as follows:

(3) *Classify the* $\mathbf{Q}$*-homology planes with logarithmic Kodaira dimension* 1*, which are obtained as the complements of irreducible plane curves.*

The scheme of the present article is as follows. In Section 1 we fix our terminology and state preliminary results without proof. In particular, the *Euclidean transformations* and the *EM-transformations* play very important roles. In Section 2 we shall state the result (Theorem 2.1) concerning the problem (1) and prove it. As seen there, such curves are classified into two types, say a curve *of the first type* and *of the second type* . In Section 3 we consider the curves of the first type and write down the defining equations as a solution to the problem (2) (cf. Theorem 3.5). In Section 4 we consider the curves of the second type. Not as in the case of the first type, the situation is more complicated and tough. We shall give the answer to the problem (2) with some additional hypotheses (cf. Theorems 4.5, 4.13 and 4.16). We make frequent use of Lemma 1.4 which is mainly due to Miyanishi and Sugie [9, Lemmas 2.15 and 2.16], though the statements given there contain some mistakes. We make the corrected statements in Lemma 1.4.

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1 Preliminaries

Let S be a smooth algebraic surface. A *smooth compactification* of S is a smooth projective surface $\overline{S}$ such that S is an open set of $\overline{S}$ and that the boundary divisor $D := \overline{S} - S$ is a divisor with simple normal crossings. A surjective morphism $\varphi : S \to B$ from a smooth algebraic surface onto a smooth algebraic curve is called an *untwisted* (resp. *twisted*) $\mathbf{C}^*$-fibration if S has a smooth compactification $\overline{S}$ of S and a $\mathbf{P}^1$-fibration $\overline{\varphi} : \overline{S} \to \overline{B}$ such that $\overline{B}$ is a smooth projective curve containing B as an open set, $\overline{\varphi}|_S = \varphi$ and $\overline{\varphi}$ has exactly two cross-sections (resp. one 2-section) contained in the boundary $\overline{S} - S$.

We shall recall the definitions of *Euclidean transformation* and *EM-transformations*, which will play very important roles in the subsequent arguments. Let V_0 be a smooth projective surface, let p_0 be a point on V_0 and let l_0 be an irreducible curve on V_0 such that p_0 is a simple point of l_0. Let d_0 and d_1 be positive integers such that $d_1 < d_0$. By the Euclidean algorithm with respect to $d_1 < d_0$, we find positive integers $d_2, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$:

$$\left\{\begin{array}{rcll} d_0 & = & q_1 d_1 + d_2 & d_2 < d_1 \\ d_1 & = & q_2 d_2 + d_3 & d_3 < d_2 \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ d_{\alpha-2} & = & q_{\alpha-1} d_{\alpha-1} + d_\alpha & d_\alpha < d_{\alpha-1} \\ d_{\alpha-1} & = & q_\alpha d_\alpha & q_\alpha > 1 \end{array}\right.$$

Set $N := \sum_{s=1}^{\alpha} q_s$. Define the infinitely near points p_i's of p_0 for $1 \leq i < N$ and the blowing-ups $\sigma_i : V_i \to V_{i-1}$ with center at p_{i-1} for $1 \leq i \leq N$ inductively as follows:

(i) p_i is an infinitely near point of order one of p_{i-1} for $1 \le i < N$.

(ii) Let $E_i := \sigma_i^{-1}(p_{i-1})$ for $1 \le i \le N$ and let $E(s,t) := E_i$ if $i = q_1 + \ldots + q_{s-1} + t$ with $1 \le s \le \alpha$ and $1 \le t \le q_s$, where we set $q_0 := 0$ and $E(0,0) := l_0$. The point p_i is an intersection point of the proper transform of $E(s-1, q_{s-1})$ on V_i and the exceptional curve $E(s,t)$ if $i = q_1 + \ldots + q_{s-1} + t$ with $1 \le s \le \alpha$ and $1 \le t \le q_s$ ($1 \le t < q_\alpha$ if $s = \alpha$).

Then a composite $\sigma := \sigma_1 \cdot \cdots \cdot \sigma_N$ is called an *Euclidean transformation* associated with the datum $\{p_0, l_0, d_0, d_1\}$ (cf. Miyanishi [6, p.92]). The weighted dual graph of Supp $(\sigma^{-1}(l_0))$ is given in Figure 1, where $E_0 := \sigma'(l_0)$ which denotes the proper transform of l_0 by σ and where we denote the proper transform of $E(s,t)$ on V_N by the same notation.

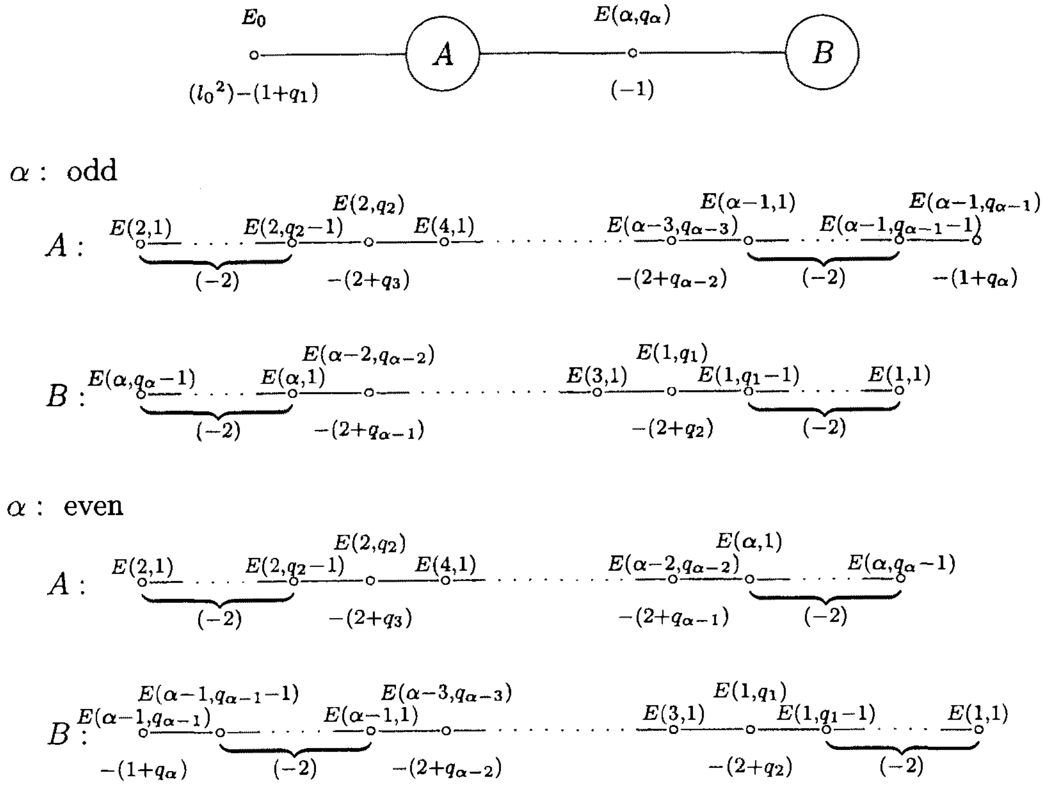


Figure 1:

Let C_0 be an irreducible curve on V_0 such that p_0 is a one-place point of C_0, let d_0 be the local intersection number $i(C_0 \cdot l_0; p_0)$ of C_0 and l_0 at p_0 and let d_1 be the multiplicity $\mathrm{mult}_{p_0}(C_0)$ of C_0 at p_0. Then $d_0 > d_1$. The proper transform $C_i := (\sigma_1 \cdots \sigma_i)'(C_0)$ passes through p_i so that $(C_i \cdot E(s,t)) = d_s$ and the intersection number of C_i with the proper transform of $E(s-1, q_{s-1})$ on V_i is $d_{s-1} - td_s$, where $i = q_1 + \ldots + q_{s-1} + t$. The smaller one of d_s and $d_{s-1} - td_s$ is the multiplicity of C_i at p_i for p_i is a one-place point of C_i. Note that the proper transform $\sigma'(C_0)$ on V_N meets the last exceptional curve $E(\alpha, q_\alpha)$ with order d_α and does not $E_0 := \sigma'(l_0)$ and other exceptional curves arising in the blowing-up process σ.

We now explain EM-transformation, which is called an (e,i)-transformation in Miyanishi [6, p.100]. Let V_0, p_0 and l_0 be the same as above. Let $r > 0$ be a positive integer. An *equi-multiplicity transformation* (or *EM-transformation*, for short) of *length* r with center at p_0 is a composite $\tau = \tau_1 \cdots \tau_r$ of blowing-ups defined as follows. For $1 \le i \le r$, $\tau_i : V_i \to V_{i-1}$ is defined inductively as the blowing-up with center at p_{i-1} and p_i is a point on $\tau_i^{-1}(p_{i-1})$ other than the intersection point $\tau_i'(\tau_{i-1}^{-1}(p_{i-2})) \cap \tau_i^{-1}(p_{i-1})$ ($\tau_1'(l_0) \cap \tau_1^{-1}(p_0)$ if $i = 1$). Let C_0 be an irreducible curve on V_0 such that p_0 is a one-place point of C_0. Suppose $d_0 := i(C_0 \cdot l_0; p_0)$ is equal to $d_1 := \mathrm{mult}_{p_0}(C_0)$. Let $\tau_1 : V_1 \to V_0$ be the blowing-up with center p_0, and set $E_1 := \tau_1^{-1}(p_0)$ and $C_1 := \tau_1'(C_0)$. Then the point $p_1 := C_1 \cap E_1$ differs from $\tau_1'(l_0) \cap E_1$. Set $d_0^{(1)} := i(C_1 \cdot E_1; p_1) = d_1$ and $d_1^{(1)} := \mathrm{mult}_{p_1}(C_1)$. Suppose $d_0^{(1)} = d_1^{(1)}$. As above, let $\tau_2 : V_2 \to V_1$ be the blowing-up with center p_1, let $E_2 := \tau_2^{-1}(p_1)$ and let $C_2 := \tau_2'(C_1)$. Then $p_2 := C_2 \cap E_2$ differs from the point $\tau_2'(E_1) \cap E_2$. Thus this process can be repeated as long as the intersection number of the proper transform of C_0 with the last exceptional curve is equal to the multiplicity of the proper transform of C_0 at the intersection point. If we perform the blowing-ups r times, the composite of r blowing-ups is an EM-transformation of length r.

We define the notion of an oscilating transformation which is to be used in Sections 3 and 4. Let V_0, l_0 and p_0 be the same as above. Let $(n_1, \cdots, n_r)$ be a sequence of positive integers. Let $\theta_1 : V_1 \to V_0$ be a composite of the n_1 successive blowing-ups with centers at p_0 and its infinitely near points lying on the proper transforms of l_0 and let p_1 be the intersection point of the last and the second last exceptional components in the process θ_1. We define the birational morphism $\theta_i : V_i \to V_{i-1}$ and the point p_i on V_i for $2 \le i \le r$

inductively as follows: Suppose that $\theta_{i-1} : V_{i-1} \to V_{i-2}$ and the point p_{i-1} on V_{i-1} are defined. Let $\theta_i : V_i \to V_{i-1}$ be a composite of the n_i successive blowing-ups with centers at p_{i-1} and its infinitely near points lying on the proper transforms of the second last exceptional component in the process θ_{i-1}. Then a composite $\theta = \theta_1 \cdots \theta_r$ is called an *oscilating transformation associated with* $(p_0, l_0; n_1, \cdots, n_r)$.

The following elementary result concerning the singular fibers of a $\mathbf{P}^1$-fibration is useful in various arguments (cf. Miyanishi [6, p.115]).

Lemma 1.1 *Let $f : V \to B$ be a $\mathbf{P}^1$-fibration on a smooth projective surface V with a smooth complete curve B. Let $F := n_1C_1 + \ldots + n_rC_r$ be a reducible singular fiber of f, where C_i is an irreducible component. Then we have :*

(1) $\gcd(n_1, \ldots, n_r) = 1$ *and* $\mathrm{Supp}\,(F) = \cup_{i=1}^r C_i$ *is connected.*

(2) *For $1 \le i \le r$, C_i is isomorphic to $\mathbf{P}^1$ and $({C_i}^2) < 0$.*

(3) *For $i \ne j$, $(C_i \cdot C_j) = 0$ or 1.*

(4) *For three distinct indices i, j and k, $C_i \cap C_j \cap C_k = \emptyset$.*

(5) *At least one of the C_i's, say C_1, is a (-1)-curve.*

(6) *If one of the n_i's, say n_1, is equal to 1, then there exists a (-1) curve among the C_i's with $2 \le i \le r$.*

The next result is a corollary of Lemma 1.1, but we encounter the situation which we can apply it to.

Lemma 1.2 *With the above notations, we suppose that*

(i) *there are two cross-sections H_1, H_2 of f,*

(ii) *there is a component H of F such that $F_{\mathrm{red}} - H$ is a disjoint union of connected components $B_1, B_2, \ldots, B_r$ with $r \ge 3$, and*

(iii) *The component H is linked to the cross-sections H_1 (resp. H_2) via a linear chain contained in B_1 (resp. B_2),*

Then each of the connected components B_i $(3 \le i \le r)$ contains a (-1) component and is contractible to a smooth point.

Proof. Suppose that either B_1 or B_2 is not contractible to a smooth point. Suppose further that some of the components $B_3, \ldots, B_r$, say B_3, is not contractible to a smooth point. After suitable contractions of the components of $B_3, \ldots, B_r$, we may assume that B_3 is not empty and that any of $B_3, \ldots, B_r$ contains no (-1) components if it is not empty. Then B_1 or B_2 contains a (-1) component, say E. Suppose that E is contained in B_1. Contarct the component E and subsequently contractible components of B_1. Suppose B_1 then becomes empty. Hence B_2 is not contractible to a smooth point by the assumption. Then, after suitable contractions of the components in B_2, we may assume that B_2 contains no (-1) component and that H is a unique (-1) component of the fiber F. But this is a contradiction because two or more different components of the same fiber meet the cross-section H_1 after the contraction of H. Suppose B_1 (as well as B_2) does not become empty after possible contractions of the components of B_1 (in B_2). Then we may assume that H is a unique (-1) component in F. This is also a contradiction because there are distinct three or more components of F meeting a (-1) component H. Next suppose that both B_1 and B_2 are contractible to smooth points. Then the component H has multiplicity one in the fiber F. Then we can contract the components $B_3, \ldots, B_r$ to smooth points. Q.E.D.

In order to look into the structures of $\mathbf{Q}$-homology planes with $\mathbf{C}^*$-fibrations, the following result is important (cf. Miyanishi and Sugie [9, Lemma 1.4]).

Lemma 1.3 *Let S be a $\mathbf{Q}$-homology plane with a $\mathbf{C}^*$-fibration $\phi : S \to B$, where B is a smooth curve. Then B is isomorphic to $\mathbf{P}^1$ or $\mathbf{A}^1$. Furthermore, the following assertions hold true:*

(1) *If B is isomorphic to $\mathbf{P}^1$ then ϕ is untwisted, every fiber of ϕ is irreducible and there is exactly one fiber, say F, such that $F_{\text{red}} \cong \mathbf{A}^1$.*

(2) *If B is isomorphic to $\mathbf{A}^1$ and ϕ is untwisted, then all fibers of ϕ are irreducible except for one singular fiber which consists of two irreducible components. If B is isomorphic to $\mathbf{A}^1$ and ϕ is twisted, all fibers are irreducible and there is exactly one fiber which is isomorphic to a multiple of $\mathbf{A}^1$.*

The following result is useful to calculate the value of $\overline{\kappa}(S)$ for a $\mathbf{Q}$-homology plane S with an untwisted $\mathbf{C}^*$-fibration onto an $\mathbf{A}^1$. This result is due to Miyanishi and Sugie [9, Lemmas 2.15 and 2.16]. The original statement of the result has some minor flaws, and the rectified statement is given as follows. The proof is easy, and we omit it.

Lemma 1.4 *Let S be a $\mathbf{Q}$-homology plane with an untwisted $\mathbf{C}^*$-fibration $\phi : S \to \mathbf{A}^1$. Then the following assertions hold true:*

(1) *ϕ has a unique reducible fiber, say G_0, which consists of two components, say $G_{0,1}$ and $G_{0,2}$. All other singular fibers of ϕ are multiples of curves isomorphic to $\mathbf{C}^*$. Let $m_{0,1}$ and $m_{0,2}$ be the multiplicities of $G_{0,1}$ and $G_{0,2}$ in G_0, respectively and let $G_i := m_i\mathbf{C}^*$ exhaust all irreducible multiple fibers of φ (if there exist such curves at all) for $1 \leq i \leq r$.*

(2) *The configuration of* $\mathrm{Supp}\,(G_0) = G_{0,1} \cup G_{0,2}$ *is described in one of the following fashions:*

1 *$G_{0,1} \cong G_{0,2} \cong \mathbf{A}^1$, and $G_{0,1}$ and $G_{0,2}$ meet in one point transversally.*

2 *$G_{0,1} \cong \mathbf{A}^1$, $G_{0,2} \cong \mathbf{C}^*$ and $G_{0,1} \cap G_{0,2} = \emptyset$.*

(3) (3-1) *In the case* **1**, *then $\overline{\kappa}(S) = 1, 0$ or $-\infty$ if and only if*

$$r - \frac{1}{\min(m_{0,1}, m_{0,2})} - \sum_{i=1}^{r} \frac{1}{m_i} > 0, \;\; = 0 \;\; or \;\; < 0, \;\; respectively.$$

(3-2) *In the case* **2**, *then $\overline{\kappa}(S) = 1, 0$ or $-\infty$ if and only if*

$$r - \frac{1}{m_{0,2}} - \sum_{i=1}^{r} \frac{1}{m_i} > 0, \;\; = 0 \;\; or \;\; < 0, \;\; respectively.$$

The following lemma is shown by a straightforward computation. So, we shall omit the proof.

Lemma 1.5 *Let d_0 and d_1 be positive integers such that $d_0 > d_1$ and $\gcd(d_0, d_1) = 1$. Let $d_2, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$ be the positive integers obtained by the Euclidean algorithm with respect to $d_1 < d_0$. Let $q'_s := q_{\alpha+1-s}$ for $1 \leq s \leq \alpha$. Define positive integers $b(s,t)$ for $1 \leq s \leq \alpha$ and $1 \leq t \leq q'_s$ as follows:*

$$\begin{array}{rcll} b(1,t) & := & 1+t & 1 \leq t \leq q'_1 \\ b(2,t) & := & b(1,q'_1) + tb(1,q'_1 - 1) & 1 \leq t \leq q'_2 \\ b(s,t) & := & b(s-1,q'_{s-1}) + tb(s-1,q'_{s-1} - 1) & 2 \leq s \leq \alpha, 1 \leq t \leq q'_s \end{array}$$

Then $b(\alpha - i, q'_{\alpha-i} - 1) = d_i$ for $0 \leq i \leq \alpha - 1$.

2 The complement of an irreducible plane curve

In this section we treat the problem (1) in Section 0 and prove the following result.

Theorem 2.1 *Let C be an irreducible plane curve with two cuspidal points and let $X := \mathbf{P}^2 - C$. Suppose $\overline{\kappa}(X) = 1$. Then there exists an irreducible linear pencil Λ on $\mathbf{P}^2$ such that the restriction of Φ_Λ onto X gives rise to an untwisted $\mathbf{C}^*$-fibration*

$$\varphi := \Phi_\Lambda|_X : X \longrightarrow \mathbf{A}^1,$$

where Φ_Λ is the rational mapping defined by Λ. More precisely, the linear pencil Λ satisfies the following properties:

(1) *Λ has two base points which are the singular points of C.*

(2) *Λ has a unique reducible member with two irreducible components, say $\overline{F_1} = m_{11}\overline{F_{11}} + m_{12}\overline{F_{12}}$, and a unique irreducible multiple member, say $\overline{F_2}$.*

(3) *C is an irreducible reduced member of Λ.*

The unique reducible member $\overline{F_1}$ produces a reducible fiber of φ, $F_1 := \overline{F_1} \cap X = m_{11}F_{11} + m_{12}F_{12}$, where $F_{1j} := \overline{F_{1j}} \cap X$ for $j = 1, 2$. Furthermore, the fiber F_1 has one of the following configurations:

1. *$F_{11} \cong F_{12} \cong \mathbf{A}^1$, and F_{11} and F_{12} meet each other in one point transversally.*

2. *$F_{11} \cong \mathbf{A}^1$, $F_{12} \cong \mathbf{C}^*$ and $F_{11} \cap F_{12} = \emptyset$.*

We say that a curve C is *of the first type* (resp. *of the second type*) if the case 1 (resp. the case 2) occurs.

Our proof consists of several steps. First of all, by Kawamata [4], there exists a $\mathbf{C}^*$-fibration φ on X. Since the base curve of φ is rational, the closures of general fibers of φ generates an irreducible linear pencil Λ on $\mathbf{P}^2$ such that $\Phi_\Lambda|_X = \varphi$. We first prove the following result.

Lemma 2.2 *The curve C is contained in a member of Λ.*

Proof. Suppose that C is not contained in any member of Λ. Let C_1 be a general member of Λ. Noting that C_1 has two places lying on C, we have the following three cases to consider:

1. C_1 meets C in only one point.

2. C_1 meets C in two smooth points.

3. C_1 meets C in one smooth point and one of the two singular points.

In the first case, let $p_1 = C_1 \cap C$. Then p_1 is a singular point of C_1 because two places of C_1 lie over the point p_1. Since p_1 moves as C_1 moves in Λ, this contradicts the second theorem of Bertini. In the second case, two general members C_1, C_2 do not meet on $\mathbf{P}^2$, which is impossible. Here note that if Λ has two base points on C then C is contained in a member of Λ. In the third case, the singular point, say p_0, is a base point of Λ. Let $\sigma : V \to \mathbf{P}^2$ be the shortest succession of blowing-ups with centers at p_0 and its infinitely near points such that the proper transform $\sigma'(\Lambda)$ of Λ by σ has no base points. Note that σ is a composite of Euclidean transformations and EM-transformations, which are uniquely determined by the general members of Λ because a general member of Λ has the point p_0 as a one-place point. Note that we may identify $\sigma^{-1}(X)$ with X. Among the boundary curves in $D := V - X$, the last exceptional curve in the process of σ, say H, and the proper transform $\sigma'(C)$ of C are the cross-sections of a $\mathbf{P}^1$-fibration defined by $\sigma'(\Lambda)$, and all other boundary components are contained in some members of $\sigma'(\Lambda)$. Thus φ is an untwisted $\mathbf{C}^*$-fibration with base curve $\mathbf{P}^1$. By Lemma 1.3 every fiber of φ is irreducible and there is exactly one fiber, say F, such that $F_{\text{red}} \cong \mathbf{A}^1$. Such a fiber exists only when H and $\sigma'(C)$ meet in one point or they are connected by exceptional components in the process σ. By looking at the configuration of the boundary D, all other fibers of φ are isomorphic to $\mathbf{C}^*$. Hence X contains a Zariski open subset U isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$. But then $\overline{\kappa}(X) \le \overline{\kappa}(U) = 0$, a contradiction to the hypothesis $\overline{\kappa}(X) = 1$. Thus the third case does not occur. Q.E.D.

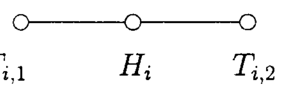
Lemma 2.3 BsΛ *consists of two singular points, say* p_1 *and* p_2, *of* C.

Proof. Since C is contained in a member of Λ by Lemma 2.2 and since any irreducible component of a $\mathbf{P}^1$-fibration is smooth by Lemma 1.1, two singular points are contained in the base locus of Λ. Q.E.D.

Let Δ be a member of Λ which contains C as an irreducible component. Let $\sigma : V \longrightarrow \mathbf{P}^2$ be the shortest succession of blowing-ups with centers at $\mathrm{Bs}\Lambda$ including their infinitely near points such that the proper transform $\sigma'(\Lambda)$ has no base points. We shall collect more informations on the construction of the process σ.

Construction of σ: For a general member G of Λ, let l_1, l_2 be the tangent lines of G at p_1, p_2, respectively. Set $d_{i,0} := i(G \cdot l_i; p_i), d_{i,1} := \mathrm{mult}_{p_i} G$ for $i = 1, 2$. Note that $d_{i,0} > d_{i,1}$. Indeed, if the equality occurs for $i = 1$ say, G is a line and Λ consists of lines. So C is a line and $\overline{\kappa}(X) = -\infty$, which is a contradiction. Note again that the point p_i is a one-place point of a general member G of the pencil Λ for $i = 1, 2$. Hence, after a succession of blowing-ups, say τ, with centers at p_i and its infinitely near points, the proper transform $\tau'(G)$ has only one point, say q_i, lying above p_i, which is, by the Bertini theorem, a base point of the proper transform $\tau'(\Lambda)$ as long as q_i is a singular point of $\tau'(G)$. This implies that the process of eliminating the base points of Λ is the process of resolving the singularities of G at the points p_i followed by the process of separating two (already resolved) general members. Hence the process of eliminating the base points of Λ is written as a composite of the Euclidean transformations and the EM-transformations applied independently at the points p_i. Let σ_i $(i = 1, 2)$ be the shortest one, which starts with the Euclidean transformation associated with the datum $\{p_i, l_i, d_{i,0}, d_{i,1}\}$ (cf. Section 1), such that $\sigma_i'(\Lambda)$ has no base points on the last exceptional curve H_i in the process σ_i. Note that σ_1 and σ_2 can be performed independently. Then a composite $\sigma = \sigma_1 \cdot \sigma_2 : V \longrightarrow \mathbf{P}^2$ is the one we require. Note that among the boundary components of $D = V - X$, H_1 and H_2 are cross-sections of $\Lambda_V := \sigma'(\Lambda)$ and all other components are contained in some members of Λ_V.

Since $\sigma^{-1}(p_i)$ $(i = 1, 2)$ is a tree consisting of H_i and two connected trees $T_{i,1}, T_{i,2}$ lying on both sides of H_i, where $T_{i,1}$ or $T_{i,2}$ might be an empty set:



The trees $T_{i,1}, T_{i,2}$ are contained in two reducible fibers. If a fiber of Λ_V containing $T_{1,1}$ has only one more component A then the closure $\overline{A}$ of A meets the cross-section H_2. Hence the multiplicity of A must be one. So, for A to be a multiple fiber, $\overline{A}$ meets one tree from the $T_{1,j}$'s and one tree from the $T_{2,j}$'s, where $j = 1, 2$. It follows from this consideration that Λ has at most two multiple members.

Let Δ be the member of Λ containing C as an irreducible component. Then either $C \subsetneqq \mathrm{Supp}(\Delta)$ or $C = \mathrm{Supp}(\Delta)$. We prove, in fact, the following result.

Lemma 2.4 *The first case does not occur. Namely, C is a unique irreducible component of Δ.*

Proof. Suppose the contrary that Δ contains another component C_1. Then φ is an untwisted $\mathbf{C}^*$-fibration on X parametrized by $\mathbf{P}^1$. Hence Lemma 1.3 says that all fibers of φ are irreducible and there exists exactly one fiber, say F_0, with $F_{0,\mathrm{red}} \cong \mathbf{A}^1$. Write $\Delta = mC + m_1C_1$. Suppose further that Δ cuts out the fiber F_0. We claim that there exist exactly two irreducible multiple members of Λ, say Δ_1 and Δ_2, such that $\Delta_1 \cap X$ and $\Delta_2 \cap X$ are the multiples of $\mathbf{C}^*$. In fact, if there exists none or only one such fiber, then X would contain a Zariski open subset U isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$. But then $\overline{\kappa}(X) \le \overline{\kappa}(U) = 0$, a contradiction to the hypothesis $\overline{\kappa}(X) = 1$. Let $\tilde{\Delta}$ be the member of Λ_V corresponding to Δ. Note that $\tilde{\Delta}$ consists of $\tilde{C} := \sigma'(C)$ and $\widetilde{C_1} := \sigma'(C_1)$, for all components of Supp $(\sigma^{-1}(p_1, p_2))\backslash(H_1 \cup H_2)$ are contained in the members of Λ_V corresponding to Δ_1 and Δ_2. Moreover, it follows that C and C_1 meet transversally in one point other than the base points p_1, p_2 and that C_1 does not pass through p_1, p_2. For H_1 and H_2 are the cross-sections of Λ_V and $\tilde{C}$ meets H_1 and H_2. This implies that C_1 does not pass through no centers of the process σ. Hence $({C_1}^2) = (\widetilde{C_1}^2) < 0$, which is a contradiction. Thus it follows that $\Delta \cap X \neq F_0$. Since $H_1 \cap H_2 = \emptyset$, the only way to obtain the singular fiber F_0 is that H_1 and H_2 are linked by some exceptional components of the process σ. This is clearly not possible. So, C is a unique irreducible component of Δ. Q.E.D.

As a consequence of Lemma 2.4, we know that φ is an untwisted $\mathbf{C}^*$-fibration parametrized by $\mathbf{A}^1$. Then Lemma 1.4 says that φ has a unique reducible fiber, say F_1, which consists of two irreducible components, say F_{11}

and F_{12}. The configuration of Supp (F_1) is described in one of the following fashions:

(1) $F_{11} \cong F_{12} \cong \mathbf{A}^1$ and $F_{11} \cap F_{12} \neq \emptyset$.

(2) $F_{11} \cong \mathbf{A}^1, F_{12} \cong \mathbf{C}^*$ and $F_{11} \cap F_{12} = \emptyset$.

In the first case (resp. the second case), we say that the curve C is *of the first type* (resp. *of the second type*). Let $\widetilde{F_1}$ be the member of Λ_V corresponding to F_1.

Suppose $\widetilde{F_1}$ contains no components of the boundary D. Then $\widetilde{F_1}$ consists of the closures C_{11} and C_{12} of F_{11} and F_{12} on V, respectively. If C is a curve of the first type, then the multiplicities of C_{11} and C_{12} in the fiber $\widetilde{F_1}$ are equal to 1. Then the Bezout theorem implies that $\deg(\overline{F_{11}}) = \deg(\overline{F_{12}}) = 1$ and that the degree of a general member of Λ is equal to 2, where $\overline{F_{11}}, \overline{F_{12}}$ are the closures of F_{11}, F_{12} on $\mathbf{P}^2$, respectively. Since C or its multiple is a member of Λ, it follows that C is a line or a conic and that $\overline{\kappa}(X) = -\infty$. This is a contradiction. If C is a curve of the second type, C_{11} and C_{12} meet each other at a point on the cross-section H_1 or H_2. This is also a contradiction.

Thus $\widetilde{F_1}$ contains some exceptional components of the process σ and Λ has at most one irreducible multiple member. If either Λ has no irreducible multiple members or Δ itself is a multiple member, say $\Delta = mC$ with $m > 1$, then X contains a Zariski open subset U isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$, which leads to a contradiction to the hypothesis $\overline{\kappa}(X) = 1$. Hence it follows that Λ has one and only one irreducible multiple member, say $\overline{F_2}$, and that C is a member of Λ, i.e., $\Delta = C$.

Thus we proved all the assertions of Theorem 2.1.

3 Case C is a curve of the first type

In this section, we consider the case where C is an irreducible plane curve of the first type and determine its defining polynomial (see Theorem 3.5). Let $F_1 = m_{11}F_{11} + m_{12}F_{12}$ be the unique reducible fiber of φ, where $F_{11} \cong F_{12} \cong \mathbf{A}^1$ and $F_{11} \cap F_{12} \neq \emptyset$. Let F_2 be a unique irreducible multiple fiber of φ. Let $\overline{F_1} = m_{11}\overline{F_{11}} + m_{12}\overline{F_{12}}$ (resp. $\overline{F_2}$) be the member of Λ corresponding to F_1 (resp. F_2), where $\overline{F_{11}}$ and $\overline{F_{12}}$ are lines on $\mathbf{P}^2$. Let $\widetilde{F_1}$ (resp. $\widetilde{F_2}$) be the member of Λ_V corresponding to $\overline{F_1}$ (resp. $\overline{F_2}$). Let C_{11}, C_{12}, C_2 be the closures of F_{11}, F_{12}, F_2 on V, respectively. Then we prove the following:

Lemma 3.1 *The configurations of $\widetilde{F_1}$ and $\widetilde{F_2}$ are linear chains.*

Proof. Note that by the construction of σ (see Section 2), all exceptional components in the process σ other than H_1 and H_2 have self-intersection number less than or equal to -2. Suppose $\widetilde{F_1}$ is not a linear chain. Then the configuration of $\widetilde{F_1} \cup H_1 \cup H_2$ is as shown in Figure 2, where there are one or more branches sprout out of the chain connecting H_1 and H_2. Note

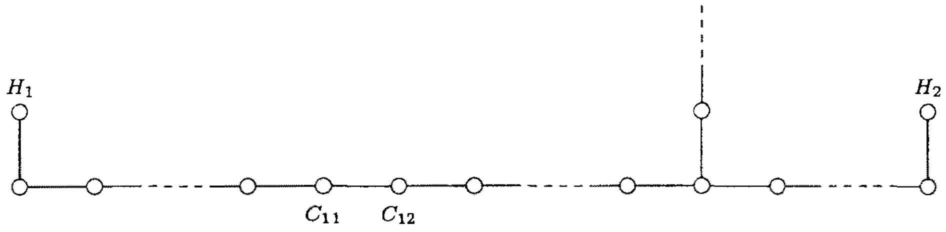


Figure 2:

that C_{11} or C_{12} is a (-1)-curve by Lemma 1.1. By successive contractions of (-1)-curves in the fiber $\widetilde{F_1}$ starting with the contraction of (-1) curve C_{11} or C_{12}, we obtain a smooth fiber of a $\mathbf{P}^1$-fibration, which is the image of the component of $\widetilde{F_1}$ intersecting the cross-section H_1. But in the course of the contraction process we encounter the configuration as shown in Figure 3, where a (-1)-curve E meets three other irreducible fiber components.

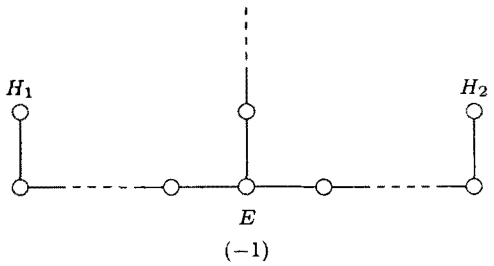


Figure 3:

This is impossible by Lemma 1.1. By a similar argument, $\widetilde{F_2}$ is also a linear chain. Q.E.D.

It follows from Lemma 3.1 that the configurations of Supp $(\sigma^{-1}(p_1))$ and Supp $(\sigma^{-1}(p_2))$ are rational linear chains. In general, Supp $(\sigma^{-1}(p_1))$ has two linear subchains on both sides of H_1, one of which is contained in the fiber $\widetilde{F_1}$ and the other in $\widetilde{F_2}$. Similar is the case of Supp $(\sigma^{-1}(p_2))$. Note that $\sigma = \sigma_1 \cdot \sigma_2$ (cf. Section 2).

Lemma 3.2 *For $i = 1, 2$, let $\mathcal{D}_i = \{p_i, l_i, d_{i,0}, d_{i,1}\}$ be the datum for the first Euclidean transformation with center p_i. Then the configuration of* Supp $(\sigma^{-1}(p_i))$ *is a linear chain if and only if σ_i is written in one of the following two fashions:*

(1) *σ_i coincides with the first Euclidean transformation,* $\gcd(d_{i,0}, d_{i,1}) = 1$ *and two general members of Λ are separated from each other after applying the first Euclidean transformation.*

(2) *$\sigma_i = \sigma_i^{(1)} \cdot \tau_i^{(1)} \cdot \sigma_i^{(2)}$, where $\sigma_i^{(j)}$ $(j = 1, 2)$ is the Euclidean transformation associated with the datum $\mathcal{D}_i^{(j)} := \{p_i^{(j)}, l_i^{(j)}, d_{i,0}^{(j)}, d_{i,1}^{(j)}\}$ $(\mathcal{D}_i^{(1)} = \mathcal{D}_i)$ and $\tau_i^{(1)}$ is an EM-transformation and where $\tau_i^{(1)}$ and $\sigma_i^{(2)}$ are possibly the identity morphism. Furthermore, $d_{i,1}^{(1)} | d_{i,0}^{(1)}$,* $\gcd(d_{i,0}^{(2)}, d_{i,1}^{(2)}) = 1$ *and two general members of Λ are separated from each other after applying the second Euclidean transformation.*

Proof. The exceptional curves arising from the Euclidean transformation with the proper transform of l_i form a linear chain. So, the first case is that the proper transform of a general member of Λ becomes smooth after the first Euclidean transformation and separated from the proper transform of a second general member. If the first Euclidean transformation $\sigma_i^{(1)}$ is followed by an EM-transformation $\tau_i^{(1)}$ (or the second Euclidean transformation $\sigma_i^{(2)}$ when $\tau_i^{(1)} = \mathrm{id}$), then the last exceptional curve of $\sigma_i^{(1)}$ must meet the proper transform of l_i. This condition is expressed as $d_{i,1}^{(1)} | d_{i,0}^{(1)}$. It is clear that there is no EM-transformation following $\sigma_i^{(2)}$. Hence $\gcd(d_{i,0}^{(2)}, d_{i,1}^{(2)}) = 1$ and two general members of Λ are separated from each other after applying the second Euclidean transformation. Q.E.D.

We shall prove, in fact, the following result.

Lemma 3.3 *Only the first case in Lemma 3.2 occurs for both σ_1 and σ_2.*

Proof. We assume that $\overline{F_{11}}$ passes through the point p_1. Then $\overline{F_{12}}$ passes through p_2. Let G be a general member of Λ. Since $\overline{F_1} = m_{11}\overline{F_{11}} + m_{12}\overline{F_{12}}$ and G meet only in the points p_1, p_2, $\overline{F_{11}}$ meets G only in one point p_1. This implies that $\overline{F_{11}}$ is the tangent line of G at p_1. Similarly, $\overline{F_{12}}$ is the tangent line of G at p_2. Assume that σ_1 is as in the second case of Lemma 3.2. Then, after performing $\sigma_1^{(1)}$, the configuration of $\sigma_1^{(1)'}(\overline{F_{11}}) \cup \mathrm{Supp}\,({\sigma_1^{(1)}}^{-1}(p_1))$ is as shown in Figure 4, where the component named A is the last exceptional

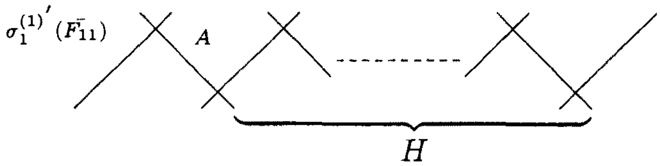


Figure 4:

curve in the process $\sigma_1^{(1)}$ and the chain on the right side of A (called H in the figure) is not empty. Let $G^{(1)}$ be the proper transform of G and let $Q_1 = G^{(1)} \cap A$. Note that the point Q_1 differs from the point $\sigma_1^{(1)'}(\overline{F_{11}}) \cap A$. We claim that the component A belongs to the member $\overline{F_1}^{(1)}$ of $\sigma_1^{(1)'}(\Lambda)$ corresponding to $\overline{F_1}$. Otherwise, A belongs to the member corresponding to $\overline{F_2}$ which gives a multiple irreducible fiber of the $\mathbf{C}^*$-fibration φ and the member $\overline{F_1}^{(1)}$ would not pass through the point Q_1, a contradiction. The components of ${\sigma_1^{(1)}}^{-1}(p_1)$ then belong to the member $\overline{F_1}^{(1)}$. If the EM-transformation $\tau_1^{(1)}$ is not the identity morphism, the same argument shows that the exceptional curves arising from $\tau_1^{(1)}$ belongs to the member corresponding to $\overline{F_1}$ and the component A would be a branching component in $\widetilde{F_1}$, which is a contradiction by Lemma 3.1. Hence $\tau_1^{(1)} = \mathrm{id}$. The second Euclidean transformation $\sigma_1^{(2)}$ is associated with the datum $\mathcal{D}_1^{(2)} = \{Q_1, A, d_{1,0}^{(2)}, d_{1,1}^{(2)}\}$, where $d_{1,0}^{(2)} = d_{1,1}^{(1)}$ and $\gcd(d_{1,0}^{(2)}, d_{1,1}^{(2)}) = 1$. We claim that $d_{1,1}^{(2)} = 1$. Suppose that $d_{1,1}^{(2)} > 1$. Then there exists a non-empty linear chain between $\sigma_2^{(2)'}(A)$ and the last exceptional curve B of $\sigma_1^{(2)}$, and the components belonging to this linear chain are contained in the member $\widetilde{F_1}$. Then the dual graph of $\widetilde{F_1}$ has a branch point, which is a contradiction to Lemma 3.1. By a similar argument, we can draw the configuaration of $\sigma_2^{-1}(p_2)$. Figure 5 is a picture of the configuration of $\widetilde{F_1} \cup \widetilde{F_2} \cup H_1 \cup H_2$ when σ_2 is in the case (2) of Lemma 3.2, where the member $\widetilde{F_1}$ is supported by the upper horizontal curves and where A_1 and A_2 are the proper transforms of the last exceptional curves of $\sigma_1^{(1)}$ and $\sigma_2^{(1)}$, respectively.

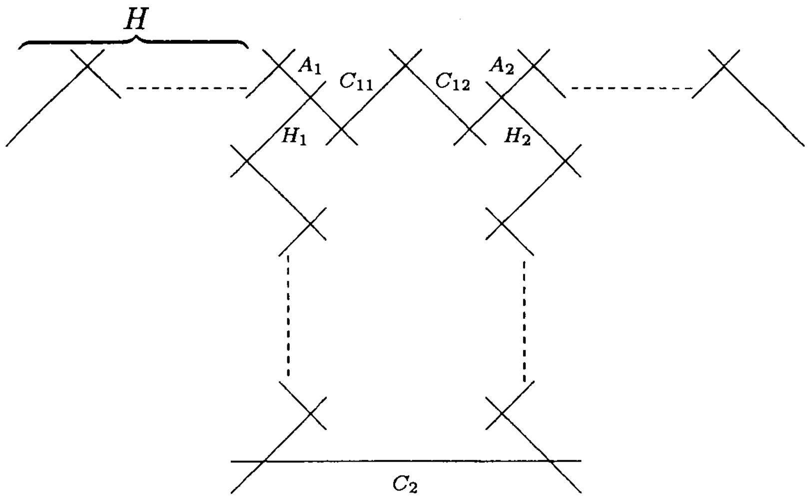


Figure 5:

Note that the member $\widetilde{F_1}$ is contracted to a smooth rational curve with either one of A_1 and A_2 left as the final image curve because A_1 and A_2 meet the cross-sections H_1 and H_2, respectively. Meanwhile, all components in the fiber $\widetilde{F_1}$ other than C_{11} and C_{12} have self-intersection number less than or equal to -2. We can obtain a smooth fiber of a $\mathbf{P}^1$-fibration from $\widetilde{F_1}$, which is the image of the component A_1 intersecting the cross-section H_1. Then the chain H is left intact and not empty. This is a contradiction. This argument applies also to the case when σ_2 is in the case (1). Q.E.D.

By Lemma 3.3, the pencil Λ is eliminated its base points by the Euclidean transformations σ_1 and σ_2 associated with the datum $\mathcal{D}_i = \{p_i, \overline{F_{1i}}, d_{i,0}, d_{i,1}\}$ such that $\gcd(d_{i,0}, d_{i,1}) = 1$ for $i = 1, 2$. Since $\overline{F_{1i}}$ is the tangent line of the general members of Λ at the point p_i, it follows that $d_{i,0} = d_0$ for $i = 1, 2$, where d_0 is the degree of a general member of Λ. We put $d_1 := d_{1,1}$. By the Euclidean algorithm with respect to $d_0 > d_1$, we obtain as in Section 1 the positive integers $d_2, \cdots, d_\alpha$ and $q_1, \cdots, q_\alpha$, where $d_\alpha = 1$.

Lemma 3.4 *With the above notations and assumptions, we have :*

(1) $\alpha \geq 2$.

(2) *With the weighted dual graph of* Supp $(\sigma^{-1}(p_1))$ *given in Figure 1,* C_{11} *meets the components* $E(2,1)$ *and* C_{12} *in the member* $\widetilde{F_1}$ *of* Λ_V, *and* C_2 *meets* $E(1,1)$ *in the member* $\widetilde{F_2}$.

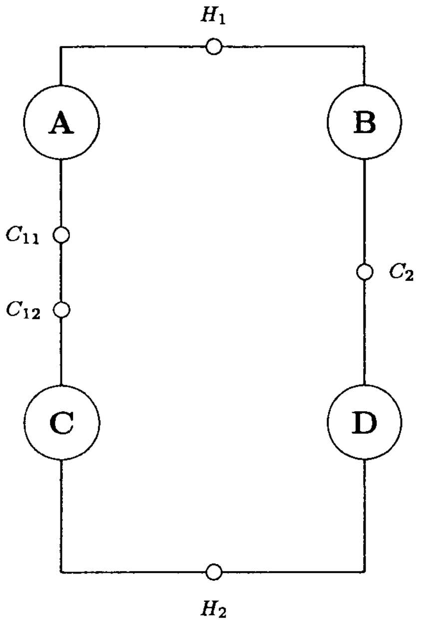
(3) *After exchanging* p_1 *and* p_2 *if necessary, we may assume that* $q_1 \geq 2$, *i.e.,* $d_0 > 2d_1$. *If* $q_1 \geq 2$, *the weighted dual graph of* $\widetilde{F_1} \cup \widetilde{F_2} \cup H_1 \cup H_2$ *is given as in Figure 6.*

Proof. (1) Suppose $\alpha = 1$. Then the component C_{11} meet the cross-section H_1. Hence the multiplicity m_{11} of F_{11} in the fiber F_1 is equal to 1. Then Lemma 1.4 (3-1)implies that $\overline{\kappa}(X) = -\infty$ because there is a unique multiple fiber F_2 in the fibration φ. This is a contradiction.

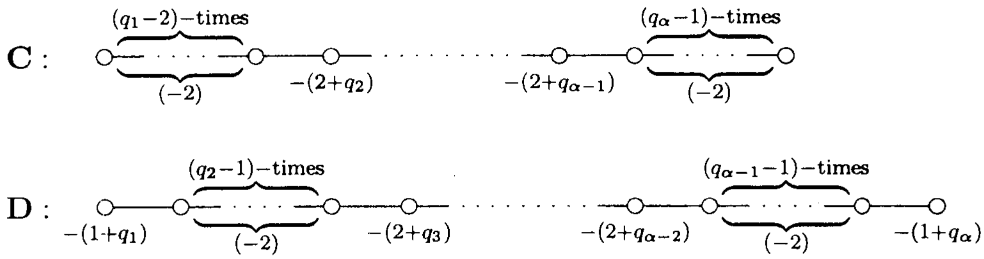
(2) Supp $(\sigma^{-1}(p_1))\backslash H_1$ consists of two connected components, one of which is contained in the member $\widetilde{F_1}$ and the other is in the member $\widetilde{F_2}$. Furthermore, one of C_{11} and C_{12} is a (-1)-curve. Since $\alpha \geq 2$, C_{11} meets the component $E(2,1)$. In the member $\widetilde{F_2}$, the component C_2, which is a unique (-1)-curve in $\widetilde{F_2}$, meets the component $E(1,1)$ or the component E_{11} meeting the cross-section H_1. But in the latter case, the contraction of E_{11} produces two components meeting the cross-section H_1. This is impossible. Hence C_2 meets the component $E(1,1)$.

(3) Suppose first that $q_1 \geq 2$. Figure 6 below then gives a picture of the weighted dual graph of $\widetilde{F_1} \cup \widetilde{F_2} \cup H_1 \cup H_2$, where A (resp. B) indicates the linear chain in Figure 1 between E_0 and $E(\alpha, q_\alpha)$ with E_0 and $E(\alpha, q_\alpha)$ excluded (resp. the linear chain between $E(\alpha, q_\alpha)$ and $E(1,1)$ with $E(\alpha, q_\alpha)$ excluded) . In the linear chains C and D, the leftmost components intersect C_{12} and C_2, respectively. By the Euclidean transformation σ_1, the proper transform C_{11} of $\overline{F_{11}}$ which is a line has self-intersection number less than or equal to -2. Hence C_{12} is a unique (-1) curve in the member $\widetilde{F_1}$. Since $\widetilde{F_1}$ is a linear chain and is contracted to a smooth member by successive contractions, the linear chain C is determined uniquely by A as indicated in Figure 6. Similarly, the linear chain D is uniquely determined by B.

Suppose next that $q_1 = 1$, i.e., $d_0 = d_1 + d_2$. Then $(C_{11}{}^2) = (\overline{F_{11}}^2) - 2 = -1$. Since $\widetilde{F_1}$ (resp. $\widetilde{F_2}$) is a linear chain and contracted to a smooth rational curve via successive contractions, which start with the contraction



α : odd



α : even

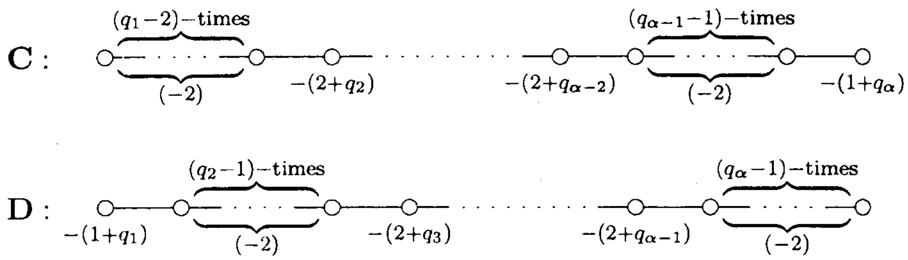


Figure 6:

of C_{11} (resp. C_2), the weighted dual graph of $\widetilde{F_1}$ (resp. $\widetilde{F_2}$) is given as in Figure 7, where we consider only the case α is even since the case α is odd is treated similarly. In Figure 7, the linear chains which are located on the right hand side of C_{12} and C_2 are contained in Supp $(\sigma^{-1}(p_2))$. By looking at the configuration of Supp $(\sigma^{-1}(p_2))$, we know that the datum $\mathcal{D}_2 = \{p_2, \overline{F_{12}}, d_0, d_{2,1}\}$ for σ_2 satisfies the following condition:

$$\frac{d_0}{d_{2,1}} = q_2 + 1 + \cfrac{1}{q_3 + \cfrac{1}{\ddots\, q_{\alpha-1} + \cfrac{1}{q_\alpha}}}.$$

Hence it follows $d_{2,1} = d_2$. Since $d_0 > 2d_2$, after exchanging the roles of p_1 and p_2, we may assume that $q_1 \geq 2$. Q.E.D.

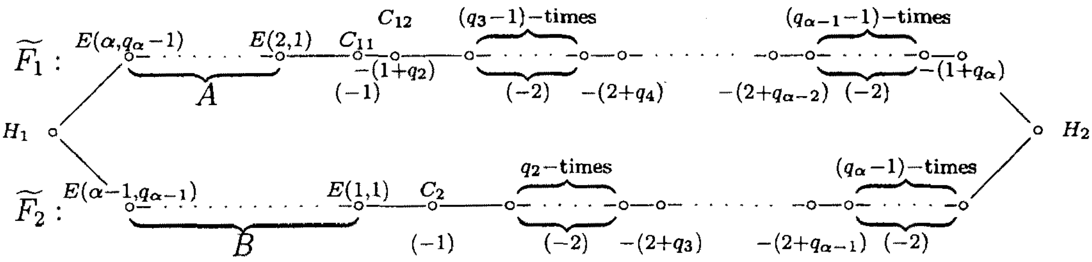


Figure 7:

With these observations in mind, we shall construct below an irreducible plane curve $C(d_0, d_1)$ of the first type with $\overline{\kappa}(\mathbf{P}^2 - C(d_0, d_1)) = 1$ for every pair of positive integers d_0 and d_1 such that $d_1 \geq 2, d_0 > 2d_1$ and $\gcd(d_0, d_1) = 1$.

Construction of $C(d_0, d_1) \subset \mathbf{P}^2$. Given a pair of positive integers d_0 and d_1 as above, we find the positive integers $d_2, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$ by the Euclidean algorithm with respect to d_0 and d_1, where $d_\alpha = 1$ (see Section 1). Let l, l_1 and l_2 be three distinct fibers of the $\mathbf{P}^1$-bundle $\Sigma_1 \to \mathbf{P}^1$, where Σ_1 is the Hirzebruch surface of degree 1. Let M_1 be the minimal section of Σ_1 and let M_2 be the cross-section such that $M_1 \cap M_2 = \emptyset$. We put $Q_1 := l_1 \cap M_2$ and $Q_2 := l_2 \cap M_2$.

Blowing up the points Q_1, Q_2 and their infinitely near points, we obtain a birational morphism $\varrho : \widetilde{V} \to \Sigma_1$ such that the configuration of $\varrho^*(l_1)$ and $\varrho^*(l_2)$ are those of $\mathrm{A}+C_{11}+C_{12}+\mathrm{C}$ and $\mathrm{B}+C_2+\mathrm{D}$ in Figure 6, respectively. Let $\widetilde{F_1} := \varrho^*(l_1), \widetilde{F_2} := \varrho^*(l_2), H_1 := \varrho^*(M_1)$ and $H_2 := \varrho' M_2$. We denote by C_{11} and C_{12} the components with self-intersection number $(-q_1)$ and (-1) in the fiber $\widetilde{F_1}$, respectively. We denote by C_2 a unique (-1)-curve in the fiber $\widetilde{F_2}$. The multiplicities of the components C_{11}, C_{12} and C_2 in the fibers $\widetilde{F_1}$ and $\widetilde{F_2}$ are $d_1, d_0 - d_1$ and d_0, respectively (see Lemma 1.5).

We can contract all components of $\widetilde{F_1} \cup \widetilde{F_2} \cup H_1 \cup H_2$ except for C_{11}, C_{12} and C_2 to the smooth points on $\mathbf{P}^2$, say p_1 and p_2. Let $\sigma : V \to \mathbf{P}^2$ be the contraction and let $C(d_0, d_1)$ be the image $\sigma(\varrho^*(l))$. Then the curves $\overline{F_{11}} := \sigma(C_{11}), \overline{F_{12}} := \sigma(C_{12})$ and $\overline{F_2} := \sigma(C_2)$ are the lines without a common point. We can take the homogeneous coordinates X, Y and Z on $\mathbf{P}^2$ such that the lines $\overline{F_{11}}, \overline{F_{12}}$ and $\overline{F_2}$ are defined by $X = 0, Y = 0$ and $Z = 0$, respectively. Let Λ be a linear pencil spanned by $d_1\overline{F_{11}} + (d_0 - d_1)\overline{F_{12}}$ and $d_0\overline{F_2}$. Then $C(d_0, d_1)$ is a member of Λ defined by $X^{d_1}Y^{d_0-d_1} + \lambda Z^{d_0} = 0$ with $\lambda \in \mathbf{C}^*$. We may take $\lambda = 1$. Meanwhile, it is clear by the construction that the complement $X := \mathbf{P}^2 - C(d_0, d_1)$ is a $\mathbf{Q}$-homology plane of the first type with an untwisted $\mathbf{C}^*$-fibration over the affine line. Note that $d_1 < d_0 - d_1$. Then Lemma 1.4 (3-1) implies that $\overline{\kappa}(X) = 1$ if and only if

$$1 - \frac{1}{d_1} - \frac{1}{d_0} > 0, \quad i.e., \quad d_1 \geq 2.$$

Conversely, a plane curve C defined by $X^{d_1}Y^{d_0-d_1} + Z^{d_0} = 0$ with $d_1 \geq 2, d_0 > 2d_1$ and $\gcd(d_0, d_1) = 1$ is a curve of the first type and its complement $\mathbf{P}^2 - C$ has log Kodaira dimension 1. Given pairs of positive integers (d_0, d_1) and (e_0, e_1) satisfying $d_1, e_1 \geq 2, d_0 > 2d_1, e_0 > 2e_1$ and $\gcd(d_0, d_1) = \gcd(e_0, e_1) = 1$, it is easy to see that $C(d_0, d_1) = C(e_0, e_1)$ up to $\mathrm{PGL}(2; \mathbf{C})$ if and only if $d_0 = e_0$ and $d_1 = e_1$.

Summarizing the above arguments and lemmas, we obtain the following theorem.

Theorem 3.5 *There exists a bijective correspondence between the set of pairs of positive integers (d_0, d_1) satisfying $d_1 \geq 2, d_0 > 2d_1$ and $\gcd(d_0, d_1) = 1$ and the set of irreducible plane curves C of the first type with $\overline{\kappa}(\mathbf{P}^2 - C) = 1$ up to $\mathrm{PGL}(2; \mathbf{C})$. The correspondence is given by $(d_0, d_1) \mapsto C(d_0, d_1) := \{X^{d_1}Y^{d_0-d_1} + Z^{d_0} = 0\}$.*

REMARK **3.6** The lowest degree case in Theorem 3.5 is $C(5,2)$. This curve is listed in Yoshihara [14] as one of the irreducible plane curves whose complement has log Kodaira dimension one.

4 Case C is a curve of the second type

In this section we shall consider a curve of the second type. We can determine a homogeneous polynomial to define such a curve only with some additional hypotheses (cf. Theorems 4.5, 4.13 and 4.16). Let C be an irreducible plane curve of the second type with $\overline{\kappa}(\mathbf{P}^2 - C) = 1$. With the same notations as in Section 3, let $F_1 = m_{11}F_{11} + m_{12}F_{12}$ be a unique reducible fiber of φ such that $F_{11} \cong \mathbf{A}^1, F_{12} \cong \mathbf{C}^*$ and $F_{11} \cap F_{12} = \emptyset$. The notations $\overline{F_1} = m_{11}\overline{F_{11}} + m_{12}\overline{F_{12}}, \overline{F_2}, \widetilde{F_1}, \widetilde{F_2}, C_{11}, C_{12}$ and C_2 are the same as at the beginning of Section 3. Among the base points of Λ, say p_1 and p_2, $\overline{F_{12}}$ and $\overline{F_2}$ pass through p_1 and p_2, while $\overline{F_{11}}$ passes through only p_1. The arguments in Lemma 3.1 implies that the configuration of $\widetilde{F_2}$ is a linear chain, but the configuration of $\widetilde{F_1}$ is not necessarily.

We write σ_1 (cf. Section 2) as

$$\sigma_1 = \sigma_1^{(1)} \cdot \tau_1^{(1)} \cdots \sigma_1^{(n-1)} \cdot \tau_1^{(n-1)} \cdot \sigma_1^{(n)} \quad \text{with } n \geq 1,$$

where $\sigma_1^{(j)}$ and $\tau_1^{(j)}$ are respectively the j-th Euclidean transformation and EM-transformation for $1 \leq j < n$ and where $\tau_1^{(j)}$ might be the identity morphism. Note that σ_1 must end with an Euclidean transformation. In fact, if it ends with an EM-transformation, then Supp $(\sigma^{-1}(p_1))\backslash H_1$ consists of one connected component, which is contained in the fiber $\widetilde{F_2}$ because H_1 is a cross-section of Λ_V and C_2(= the closure of F_2 in V) is a component of $\widetilde{F_2}$ with multiplicity ≥ 2. If Supp $(\sigma^{-1}(p_1))\backslash H_1$ is contained in $\widetilde{F_2}$ then C_{11} and C_{12} would meet each other at the point on the cross-section H_1. This is a contradiction.

First af all we prove the following:

Lemma 4.1 *The curve C_{11} is a (-1)-curve.*

Proof. The configuration of the fiber $\widetilde{F_1}$ in a simplified form is given in Figure 8. Note that C_{11} is the end component of a brached linear chain which does not contain C_{12}. Suppose C_{11} is not a (-1)-curve. Then C_{12} is

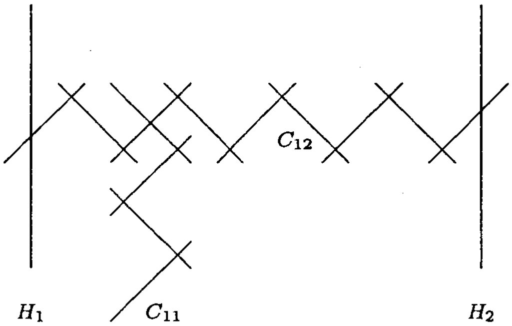


Figure 8:

a unique (-1)-curve in the fiber $\widetilde{F_1}$ and the contraction process to make $\widetilde{F_1}$ smooth starts with the contraction of C_{12}. In the course of the successive contractions, we have a (-1) component meeting at least three components of the fiber or two components of the fiber plus a cross-section. This is a contradiction by Lemma 1.1. Q.E.D.

Lemma 4.2 *The configuration of* Supp $(\sigma^{-1}(p_2))$ *is a linear chain.*

Proof. Assume to the contrary that there exists a branch component G in Supp$(\sigma^{-1}(p_2))$ from which three or more other components of Supp$(\sigma^{-1}(p_2))$ sprout out. By Lemma 3.1, G with the adjacent components are included in the fiber $\widetilde{F_1}$. Then the configuration of the fiber $\widetilde{F_1}$ in a simplified form is given in Figure 9, where the component denoted by S (resp. T) meets the cross-section H_1 (resp. H_2). Note that there are two or more branches sprouting from the chain connecting the components S and T. Then the successive contractions to make the fiber smooth which start with the contraction of C_{11} or C_{12} will produce a (-1) curve with three or more components intersecting it. This is a contradiction. Q.E.D.

In the rest of this section, we shall assume the following condition:

(#) $\overline{F_{11}}$ *is a line and* C_{12} *is a* (-1)*-curve.*

Then $\overline{F_{11}}$ is the tangent line of a general member of Λ at the point p_1. We take a system of homogeneous coordinates (X, Y, Z) on $\mathbf{P}^2$ so that $p_1 =$

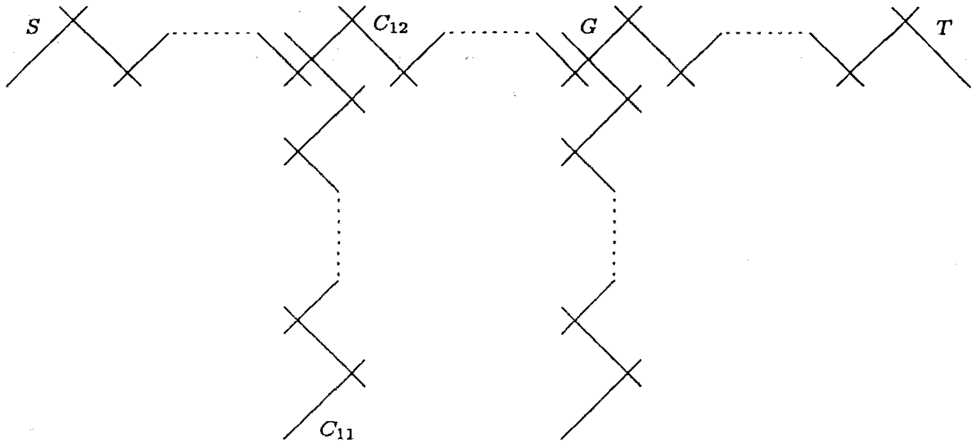


Figure 9:

$(0:1:0), p_2 = (1:0:0)$ and the line $\overline{F_{11}}$ and the tangent line of the general members of Λ at p_2, say l_2 are defined respectively by $X = 0$ and $Y = 0$. Write $\mathbf{P}^2 - \overline{F_{11}} = \mathbf{A}^2 = \operatorname{Spec} \mathbf{C}[y,z]$ and let $\iota : \mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ be the canonical open immersion as the complement of the line $\overline{F_{11}}$, where $y := Y/X$ and $z := Z/X$. Let $C^\circ := C - \{p_1\}$ and let f be an irreducible polynomial of $\mathbf{C}[y,z]$ which defines C° in $\mathbf{A}^2$. Clearly the polynomial f determines a homogeneous polynomial which defines C.

Suppose that σ_1 consists of a single Euclidean transformation, i.e., $\sigma_1 = \sigma_1^{(1)}$ in the notation at the beginning of this section, which is associated with the datum $\mathcal{D}_1 := \{p_1, \overline{F_{11}}, d_0, d_1\}$, where $d_0 := i(C \cdot \overline{F_{11}}; p_1)$ =(the degree of C) and $d_1 := \operatorname{mult}_{p_1}(C)$. Let $d_2, \cdots, d_\alpha = 1$ and $q_1, \cdots, q_\alpha$ be positive integers obtained by the Euclidean algorithm with respect to $d_0 > d_1$. Then the dual graph of $\operatorname{Supp}\sigma_1^{-1}(p_1)$ is a linear chain $A + H_1 + B$, where $H_1 = E(\alpha, q_\alpha)$ by the notation of Section 1 and where A and B are linear chains. In particular, A is the same as given in Figure 1. Since the dual graphes of $\widetilde{F}_2$ and $\operatorname{Supp}\sigma_2^{-1}(p_2)$ are linear chains, we write them as $B + C_2 + D$ and $D + H_2 + E$, respectively, where D and E are the linear chains. Since C_2 is a unique (-1) curve in $\widetilde{F}_2$, the linear chain B determines the linear chains D and then E successively. But E is not uniquely determined by D. In fact, there is some ambiguity depending on whether the last contraction occurs on the chain D or E when one contracts $D + H_2 + E$ to the smooth point p_2. If the last contraction occurs on the chain E, then the dual graph of E is given as in Figure 10. If it occurs on the chain D, the dual graph of E is the same figure with the part F deleted off.

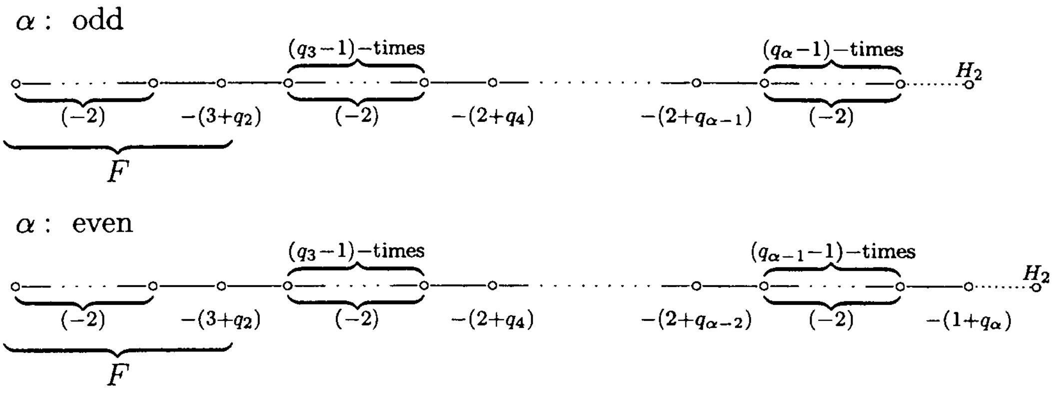


Figure 10:

Now $\operatorname{Supp} \widetilde{F_1}$ is a union $A + C_{11} + C_{12} + E$. In fact, we have the following result:

Lemma 4.3 *With the above assumptions and notations, we have:*

(1) $\alpha > 2$.

(2) *the dual graph of* $\widetilde{F_1}$ *is determined as given in Figure 12.*

Proof. (1) Suppose first $\alpha = 1$. Then $A = \emptyset$, and both C_{11} and C_{12} intersect the cross-section H_1. This is impossible. Suppose next that $\alpha = 2$. Since $\overline{F_{11}}$ is the tangent line of a general member of Λ at p_1, the component C_{11} intersects $E(2,1)$ (see Figure 1) and $({C_{11}}^2) = 1 - (1 + q_1)$. Since C_{11} is a (-1)-curve by Lemma 4.1, we have $q_1 = 1$. On the other hand, in the fiber $\widetilde{F_2}$, the unique (-1) component C_2 meets $E(1,1)$ or the terminal component of B which intersects the cross-section H_1. But the latter case leads clearly to a contradiction to Lemma 1.1. The dual graph of the fiber $\widetilde{F_1}$ is given as in Figure 11.

In order to obtain Figure 11, note that C_{12} is a (-1) curve by the hypothesis (#) and connected to some component between $E(2,1)$ and $E(2, q_2 - 1)$,

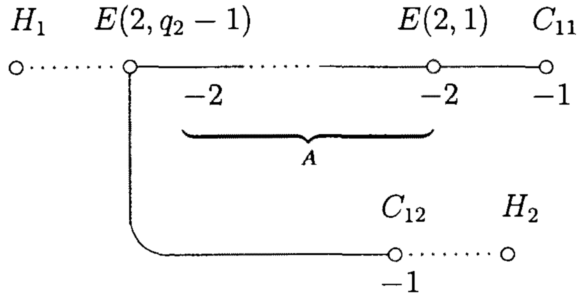


Figure 11:

say $E(2, r)$. Then $C_{12} + E(2, r) + E(2, r-1) + \cdots + E(2, 1) + C_{11}$ supports the fiber $\widetilde{F_1}$. Hence $r = q_2 - 1$ and the part E between C_{12} and H_2 is void. Then the multiplicity of C_{12} is one and Lemma 1.4 (3-2) implies $\overline{\kappa}(\mathbf{P}^2 - C) = -\infty$, which is a contradiction. Hence $\alpha > 2$.

(2) In the dual graph of the fiber $\widetilde{F_1}$, the component C_{12} intersects some component of the chain A. Since C_{11} is a (-1) curve, one can contract $C_{11}, E(2, 1), \cdots, E(2, q_2 - 1)$ in this order. After this contraction the component $E(2, q_2)$ has self-intersection number $-(1 + q_3) \leq -2$. Hence $E(2, q_2)$ is contractible after the component C_{12} is contracted. So, C_{12} intersects the component $E(2, q_2)$. Since the contraction to bring the fiber $\widetilde{F_1}$ down to a smooth rational curve does not allow a branching (-1) component, i.e., a (-1) component meeting three other components, we can show that C_{12} intersects the end component of E which is not the component meeting H_2. Hence the dual graph is as given in Figure 12. Q.E.D.

We can construct the surface V and the $\mathbf{P}^1$-fibration with the specific singular fibers $\widetilde{F_1}$ and $\widetilde{F_2}$ in the following fashion: Let Σ_1 be a Hirzebruch surface of degree one. Let l_1 and l_2 be distinct two fibers of its $\mathbf{P}^1$-fibration $\pi : \Sigma_1 \to \mathbf{P}$, let M_1 be the minimal section and let M_2 be the cross-section such that $M_1 \cap M_2 = \emptyset$. Put $Q_i := l_i \cap M_2$ for $i = 1, 2$. Let $\theta_0 : V_0 \to \Sigma_1$ be the blowing-ups with centers at Q_1 and Q_2, and let $Q_i' := l_i' \cap \theta_0^{-1}(Q_i)$ for $i = 1, 2$, where $l_i' = \theta_0'(l_i)$. We perform the oscillating transformations θ_1 associated with $(Q_1', G; q_\alpha, \cdots, q_4, q_3 - 1)$ and $(Q_2', H; q_\alpha, \cdots, q_2)$ independently (cf. Section 1), where $(G, H) = (\theta_0^{-1}(Q_1), l_2')$ if α is even and $(G, H) = (l_1', \theta_0^{-1}(Q_2))$

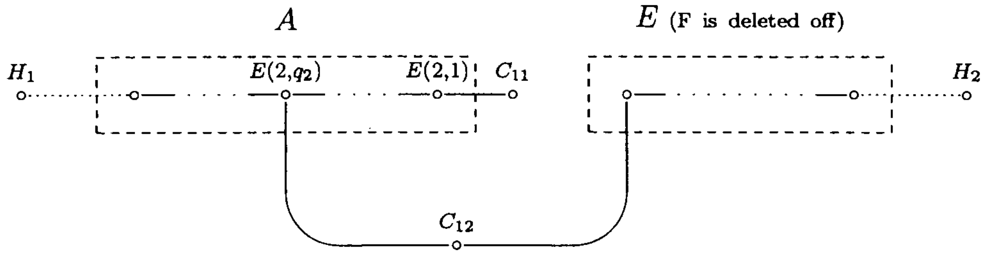


Figure 12:

if α is odd, and denote by $\theta = \theta_0 \cdot \theta_1$. Let R be the component in $\theta^*(l_1)$ with self-intersection number $-(1+q_3)$ and let Q_1'' be a point of R not lying on the other components. Let ξ be an EM-transformation of length q_2 which starts with the blowing-up with center Q_1''. Set $\varrho := \theta \cdot \xi : V \longrightarrow \Sigma_1$. Then the fibers $\varrho^*(l_1)$ and $\varrho^*(l_2)$ have respectively the same configurations as the fibers $\widetilde{F_1}$ and $\widetilde{F_2}$. Furthermore, the proper transforms of M_1 and M_2 are (-1) curves. So, we find a birational morphism $\sigma : V \longrightarrow \mathbf{P}^2$.

By the above construction and Lemma 1.5, we can show that the multiplicities of the components C_{11}, C_{12} and C_2 in the fibers $\widetilde{F_1}$ and $\widetilde{F_2}$ are d_3, d_2 and d_0, respectively. Hence the linear pencil Λ is spanned by $d_3\overline{F_{11}} + d_2\overline{F_{12}}$ and $d_0\overline{F_2}$. Note that C is an irreducible and reduced member of Λ (cf. Theorem 2.1). Since $\deg C = d_0$, it follows that

$$d_0 = d_3 + d_2 \deg \overline{F_{12}} = d_0 \deg \overline{F_2},$$

whence we know that $\deg \overline{F_{12}} = q_2 + 1$ as $q_1 = 1$ (see the proof of lemma 4.3) and that $\overline{F_2}$ is a line. Set $F_{12}^\circ := \overline{F_{12}} - \{p_1\}$ and $F_2^\circ := \overline{F_2} - \{p_1\}$.

Lemma 4.4 *With the notations as above, the curves F_{12}° and F_2° are isomorphic to the affine line. Moreover, they intersect each other in the point p_2 transversally.*

Proof. Note that C_{12} and C_2 meet the end components of $\operatorname{Supp} \sigma^{-1}(p_2)$ which is a linear chain (cf. Lemma 4.2). By successive contractions of the components in $\operatorname{Supp} \sigma^{-1}(p_2)$ which starts with H_2, it is clear that the images of C_{12} and C_2 intersect each other transversally, so $\overline{F_{12}}$ and $\overline{F_2}$ intersect in the point p_2 transversally. It is then easy to show the assertion of the lemma. Q.E.D.

We may assume that the line $\overline{F_2}$ is defined by $Z = 0$ with respect to the homogeneous coordinates (X, Y, Z) fixed after the proof of Lemma 4.2. Let f_{12} be an irreducible polynomial in $\mathbf{C}[y, z]$ to define F_{12}° on $\mathbf{A}^2 = \mathbf{P}^2 - \overline{F_{11}}$. The curve F_2° is defined by $f_2 = z$. Since the curves F_{12}° and F_2° are two affine lines intersecting each other in a point p_2 transversally (Lemma 4.4), we have $\mathbf{C}[f_{12}, f_2] = \mathbf{C}[y, z]$ (see Miyanishi [6]). Hence f_{12} is written as

$$f_{12} = cy + g(z),$$

where $c \neq 0$ and $g(z)$ is a polynomial of degree q_2+1 because $\deg \overline{F_{12}} = q_2+1$.

As a consequence of the above arguments, we obtain the following theorem.

Theorem 4.5 *Suppose that σ_1 consists of a single Euclidean transformation. Then $C^{\circ} := C - \{p_1\}$ is defined by a polynomial f in $\mathbf{P}^2 - \overline{F_{11}} = \operatorname{Spec} \mathbf{C}[y, z]$ of the following form:*

$$f = (cy + g(z))^{d_2} + \lambda z^{d_0},$$

where $c, \lambda \in \mathbf{C}^*$ and $\deg g(z) = q_2 + 1$.

From now on, we assume that σ_1 does not end with a single Euclidean transformation. Let $\mathcal{D}_1^{(j)} := \{p_1^{(j)}, l_1^{(j)}, d_0^{(j)}, d_1^{(j)}\}$ be the datum of $\sigma_1^{(j)}$ for $1 \leq j \leq n$ (see the notations at the beginning of this section). Let $d_2^{(j)}, \cdots, d_{\alpha_j}^{(j)}$ and $q_1^{(j)}, \cdots, q_{\alpha_j}^{(j)}$ be the positive integers obtained by the Euclidean algorithm with respect to $d_0^{(j)} > d_1^{(j)}$. Let $E^{(j)}(s, t)$ be the proper transform on V of the exceptional component arising from the $(q_1^{(j)} + \ldots + q_{s-1}^{(j)} + t)$-th blowing-up in $\sigma_1^{(j)}$ for $1 \leq s \leq \alpha_j$ and $1 \leq t \leq q_s^{(j)}$. Let r_j be the length of the j-th EM-transformation $\tau_1^{(j)}$ and let $E^{(j)}(l)$ be the proper transform on V of the exceptional component from the l-th blowing-up in $\tau_1^{(j)}$ for $1 \leq l \leq r_j$. To simplify the notations, we put $d_0 := d_0^{(n)}$ and $d_1 := d_1^{(n)}$ for the last Euclidean transformation. Similarly, we let $d_2, \cdots, d_\alpha$ and $q_1, \cdots, q_\alpha$ be positive integers obtained from $d_0 > d_1$, where $d_\alpha = 1$. We also put $E(s, t) := E^{(n)}(s, t)$. We prove the following result:

Lemma 4.6 *With the assumptions as above, all the exceptional components on V arising from $\sigma_1^{(j)}$ and $\tau_1^{(j)}$ for $1 \leq j < n$ are contained in the fiber $\widetilde{F_1}$.*

Proof. After the first Euclidean transformation $\sigma_1^{(1)}$, let E'' be the last exceptional component in $\sigma_1^{(1)}$. The proper transform G' by $\sigma_1^{(1)}$ of a general member G of Λ intersects only E'', among the components in $\mathrm{Supp}\sigma_1^{(1)^{-1}}(p_1)$, at a base point, say p_1', of $\sigma_1^{(1)'}(\Lambda)$. The member $\overline{F_1'}$ corresponding to $\overline{F_1}$ contains E''. In fact, the proper transform of $\overline{F_{11}}$ is separated from G' because $\overline{F_{11}}$ is the tangent line of G and some component of $\overline{F_1'}$ passes through the point p_1'. Hence the connectedness of $\overline{F_1'}$ implies that E'' as well as all the other exceptional components in $\mathrm{Supp}\,\sigma_1^{(1)^{-1}}(p_1)$ are contained in $\overline{F_1'}$. By the same argumemt, we can show that all the components on V arising from $\sigma_1^{(j)}$ and $\tau_1^{(j)}$ for $1 \le j < n$ are contained in the member $\widetilde{F_1}$. Q.E.D.

Among the components in $\mathrm{Supp}\sigma^{-1}(p_1)\backslash H_1$, the member $\widetilde{F_2}$ of Λ_V contains the components $E(s,t)$ with s odd (see the argument of Lemma 4.6). Since the dual graph of $\mathrm{Supp}\,\sigma^{-1}(p_2)$ is a linear chain (Lemma 4.2), it is written as $D + H_2 + E'$, where the part E' is a linear chain contained in the fiber $\widetilde{F_1}$. We prove the following result concerning the process σ_2.

Lemma 4.7 *With the assumptions as above, we have the following:*

(1) *The dual graph of the fiber $\widetilde{F_2}$ is the same as $B+C_2+D$ given in Figure 6, where B consists of $E(s,t)$ with s odd.*

(2) *If the last contraction to bring $D + H_2 + E'$ to a smooth point p_2 occurs on E' (resp. D) then the dual graph of the linear chain E' is given in Figure 13 (resp. Figure 13 with the part F' deleted off), where we treat the case $q_1 > 1$. The figure for the case $q_1 = 1$ is the same as in Figure 10, where the part F should be replaced by F'.*

(3) *The linear chain E' is not empty.*

Proof. (1) Since $\widetilde{F_2}$ is a linear chain, the assertion of (1) is obtained by the same argument as in the proof of Lemma 3.4.

(2) The assertion is easy to prove.

(3) Suppose that the linear chain E' is empty. Then the component C_{12} meets the cross-section H_2, so the multiplicity of C_{12} in the fiber $\widetilde{F_1}$ is one. But we then have $\overline{\kappa}(X) = -\infty$ by Lemma 1.4. This is a contradiction to the assumption $\overline{\kappa}(X) = 1$. Q.E.D.

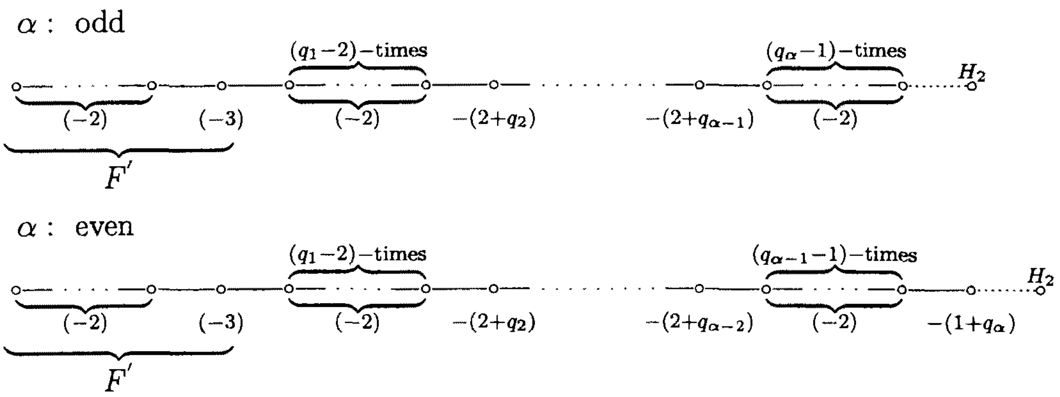


Figure 13:

Note that the component C_{12} meets the end component of E' which locates on the opposite side of H_2. For otherwise the contraction of C_{12} and subsequently contractible components would produce a (-1)-curve meeting three other components in a degenerate $\mathbf{P}^1$-fiber.

Let A' be a tree in the fiber $\widetilde{F_1}$ consisting of the exceptional components from σ_1. Then A' is written as

$$A' = B' + B_1 + B_2 + B_3,$$

where B' is the last exceptional component from $\sigma_1^{(n-1)}$,B_1 is a tree consisting of $E^{(n-1)}(s,t)$ with s even and the exceptional components arising from $\sigma_1^{(j)}$ and $\tau_1^{(j)}$ for $1 \leq j < n-1$ (if any), B_2 is a linear chain consisting of $E^{(n-1)}(s,t)$ with s odd and B_3 is a linear chain consisting of the components from $\tau_1^{(n-1)}$ and $E(s,t) := E^{(n)}(s,t)$ with s even.

Now we can specify the intermediate transformations $\sigma_1^{(j)}$ and $\tau_1^{(j)}$ for $1 \leq j < n$. Namely, we have:

Lemma 4.8 *The following assertions hold:*

(1) *For $1 \leq j < n$, one of the following two cases occurs for the datum $\mathcal{D}_1^{(j)}$ of $\sigma_1^{(j)}$:*

 (i) $d_0^{(j)} = 2d_1^{(j)}$.

 (ii) $d_0^{(j)} = d_1^{(j)} + d_2^{(j)}$ *and* $d_2^{(j)} \mid d_1^{(j)}$.

(2) *For* $1 \leq j < n-1$, *the length* r_j *of the* j*-th EM-transformation* $\tau_1^{(j)}$ *is determined by the foregoing* $\sigma_1^{(j)}$ *as follows:*

(iii) *In the case* (i) *above,* $r_j = 1$.

(iv) *In the case* (ii) *above,* $r_j = d_1^{(j)}/d_2^{(j)}$.

Proof. Let H be the component in A' meeting the component C_{12} and let L be a linear chain in A' connecting the cross-section H_1 and H with H included. By Lemma 1.2, the component H is chosen in such a way that every branch sprouting out of the linear chain $L + C_{12} + E'$ in $\widetilde{F_1}$ is contractible to a smooth point. Suppose that the component H is contained in $B' + B_1$. Then choose the component B' as the component H in Lemma 1.2. It says that the linear chain B_2 which sprouts from B' is contracted. Since B_2 contains no (-1) curves, this is impossible. Hence H is contained in $B_2 + B_3$. Furthermore, the maximal connected part B of $\widetilde{F_1}$ which branches out of the linear chain $L + C_{12} + E'$ and contains $C_{11} + B_1$ is contractible.

Suppose now that $\alpha_1 \geq 3$. Note that since $\overline{F_{11}}$ is the tangent line of a general member of Λ at p_1, C_{11} meets $E^{(1)}(2,1)$. After the contraction of the components $C_{11}, E^{(1)}(2,1), \cdots, E^{(1)}(2, q_2^{(1)}-1)$ in this order, the self-intersection number of $E^{(1)}(2, q_2^{(1)})$ then remains less than or equal to -2 and one cannot proceed further, which is a contradiction. So, $\alpha_1 \leq 2$. In the case $\alpha_1 = 1$ (resp. $\alpha_1 = 2$), we have $q_1^{(1)} = 2$, i.e., $d_0^{(1)} = 2d_1^{(1)}$ (resp. $q_1^{(1)} = 1$, i.e., $d_0^{(1)} = d_1^{(1)} + d_2^{(1)}$) and C_{11} meets $E^{(1)}(1,2)$ (resp. $E^{(1)}(2,1)$) because $({C_{11}}^2) = -1$ (Lemma 4.1). Furthermore, after the contraction of $C_{11}, E^{(1)}(1,2)$ (resp. $C_{11}, E^{(1)}(2,1), \cdots, E^{(1)}(2, q_2^{(1)})$), the self-intersection number of $E^{(1)}(1,1)$ is -1 (resp. $-q_2^{(1)}$), hence we know that the length r_1 of $\tau_1^{(1)}$ is 1 (resp. $q_2^{(1)} = d_1^{(1)}/d_2^{(1)}$) because of the contractibility of the branch B. Successively, when we contract the components $E^{(1)}(1), \ldots, E^{(1)}(r_1 - 1), E^{(1)}(1,1)$, the image of $E^{(1)}(r_1)$ must be a (-1) curve in order that the part B gets contractible. Hence $E^{(1)}(r_1)$ has self-intersection number -3 in the graph B_1. This implies that two points lying on $E^{(1)}(r_1)$ (one is infinitely near to the other) are blown-up in the process $\sigma_1^{(2)}$. This observation on $\sigma_1^{(1)}$ and $\tau_1^{(1)}$ and the contractibility of the part B imply either $\alpha_2 = 1, d_0^{(2)} = 2d_1^{(2)}$ and $r_2 = 1$, or $\alpha_2 = 2, d_0^{(2)} = d_1^{(2)} + d_2^{(2)}$ and $r_2 = d_1^{(2)}/d_2^{(2)}$. Successively, we can apply the same argument to $\sigma_1^{(j)}$ and $\tau_1^{(j)}$. Thus we have shown the assertions of (1) and (2). Q.E.D.

As shown in the proof of Lemma 4.8, the component H meeting the (-1) component C_{12} is contained in $B_2 + B_3$. By Lemma 4.8, B_2 consists only of the single component $E^{(n-1)}(1,1)$. We consider first the case where H is contained in B_3, that is, H is one of the components $E^{(n-1)}(l)$ with $1 \leq l \leq r_{n-1}$ and $E(s,t)$ with s even. Then we have the following result, where we use the simplified notations $q_i := q_i^{(n)}$, $\alpha := \alpha^{(n)}$, $E(s,t) := E^{(n)}(s,t)$ etc.

Lemma 4.9 *Suppose that H is contained in B_3. Then the following assertions hold:*

(1) *The component H in A' meeting C_{12} is determined in the following way according to the value of q_1.*

  (i) *If $q_1 = 1$, we have $\alpha > 2$ and H is equal to the component $E(2, q_2)$.*

  (ii) *If $q_1 > 1$, the curve H is equal to the component $E^{(n-1)}(r_{n-1})$.*

(2) *In both of the above cases* (i) *and* (ii)*, the length r_{n-1} of $\tau_1^{(n-1)}$ is determined by the foregoing $\sigma_1^{(n-1)}$ as follows:*

  (iii) *If $\alpha_{n-1} = 1$, we have $r_{n-1} = 1$.*

  (iv) *If $\alpha_{n-1} = 2$, we have $r_{n-1} = d_1^{(n-1)}/d_2^{(n-1)}$.*

(3) *If $q_1 = 1$, the part F in E' (see Figure 10) is empty, and if $q_1 > 1$, the part F' in E' (see Figure 13) is empty.*

Proof. Note that the component H is not a (-2) component $E^{(n-1)}(l)$ for $1 \leq l < r_{n-1}$ (if any). For otherwise, the contraction of the (-1) component C_{12} would produce a (-1) curve meeting three other components. Hence H is either $E^{(n-1)}(r_{n-1})$ or one of the $E(s,t)$ with s even. Let B be the maximal connected part which branches out of the linear chain $L + C_{12} + E'$ (see the proof of Lemma 4.8) and contains

$$C_{11} + B_1 + B_2 + B' + E^{(n-1)}(1) + \ldots + E^{(n-1)}(r_{n-1} - 1),$$

where $B_2 = E^{(n-1)}(1,1)$. Then B is contracted to a smooth point by Lemma 1.2.

As seen in the proof of Lemma 4.8, the part $C_{11} + B_1$ of B is contracted. Since the self-intersection number of $E^{(n-1)}(1,1)$ is -2 if $\alpha_{n-1} = 1$

(resp. $-(1+q_2^{(n-1)})$ if $\alpha_{n-1}=2$), the length r_{n-1} of $\tau_1^{(n-1)}$ is 1 (resp. $q_2^{(n-1)}=d_1^{(n-1)}/d_2^{(n-1)}$) by the contractibility of B. When we contract the components $B', E^{(n-1)}(1), \cdots, E^{(n-1)}(r_{n-1}-1), E^{(n-1)}(1,1)$ in this order after the contraction of $C_{11}+B_1$, the image of $E^{(n-1)}(r_{n-1})$ has self-intersection number $-q_1$. In the case $q_1>1$, the component H meeting C_{12} is equal to $E^{(n-1)}(r_{n-1})$. For otherwise, a linear chain connecting H and $E^{(n-1)}(r_{n-1})$ with H excluded cannot be contractd. This is a contradiction to Lemma 1.2. Meanwhile, in the case $q_1=1$, the image of $E^{(n-1)}(r_{n-1})$ is a (-1) curve after the above contraction. Note that we then have $\alpha>2$. Indeed, it is clear $\alpha>1$ because $q_1=1$. Suppose that $\alpha=2$. Then the remaining components of B_3 after $E^{(n-1)}(r_{n-1})$ are all (-2) components and we can contract all the components of $C_{11}+A'$ to a smooth point. Hence the (-1) curve C_{12} meets the last component $E(2,q_2-1)$ in A'. Then the part E' is an empty set, which is a contradiction by Lemma 4.7. Thus we have $\alpha>2$. When we contract the component $E^{(n-1)}(r_{n-1}), E(2,1), \cdots, E(2,q_2-1)$ in this order, the self-intersection number of $E(2,q_2)$ remains less than or equal to -2. Therefore we know that the component H is equal to $E(2,q_2)$ by Lemma 1.2. Thus we proved the assertions of the lemma. The last assertion (3) follows easily if one links $C_{12}+E'$ to the component H as indicated in the assertion (1) and considers the contraction of H after the contractions of the previous part including $B_1+B'+B_2$, C_{12} and subsequently contractible components in E'. Q.E.D.

We consider next the case where H is equal to the component $E^{(n-1)}(1,1)$. Note that B_2 consists only of $E^{(n-1)}(1,1)$ by Lemma 4.8. Then we have the following result:

Lemma 4.10 *Suppose that the component H which intersects C_{12} is $E^{(n-1)}(1,1)$. Then the following assertions hold:*

(1) *The length r_{n-1} of $\tau_1^{(n-1)}$ is determined as follows:*

 (i) *If $\alpha_{n-1}=1$, $r_{n-1}=0$.*

 (ii) *If $\alpha_{n-1}=2$, $r_{n-1}<q_2^{(n-1)}=d_1^{(n-1)}/d_2^{(n-1)}$.*

(2) *The number of (-2) curves contained in the part F' (see Figure 10 if $q_1=1$ or Figure 13 if $q_1>1$) is given as follows:*

 (iii) *If $\alpha_{n-1}=1$, the number of the (-2) components in F' is zero.*

(iv) *If* $\alpha_{n-1} = 2$, *the number of the* (-2) *components in* F' *is equal to* $q_2^{(n-1)} - (r_{n-1} + 1)$.

Proof. Let L be a linear chain in A' connecting the cross-section H_1 and $E^{(n-1)}(1,1)$, i.e., L is

$$E^{(n-1)}(1,1)+B'+E^{(n-1)}(1)+\ldots+E^{(n-1)}(r_{n-1})+\ \{\text{all } E(s,t)\text{'s with } s \text{ even}\}.$$

Then the connected part $C_{11} + B_1$ in the fiber $\widetilde{F_1}$ sprouts out of the chain $L + C_{12} + E'$, so it is contractable by Lemma 1.2. Note that $\alpha_{n-1} \leq 2$ by Lemma 4.8. By successive contractions of the components $B', E^{(n-1)}(1), \cdots, E^{(n-1)}(r_{n-1} - 1)$ (if any), which follow after the contraction of the part $C_{11} + B_1$, the image of $E^{(n-1)}(1,1)$ has self-intersection number $-2 + r_{n-1}$ if $\alpha_{n-1} = 1$ (resp. $-(1 + q_2^{(n-1)}) + r_{n-1}$ if $\alpha_{n-1} = 2$), hence the length r_{n-1} of $\tau_1^{(n-1)}$ is zero (resp. smaller than $q_2^{(n-1)}$) because C_{12} and $E^{(n-1)}(1,1)$ are the next components to be contracted in this order to make the whole fiber $\widetilde{F_1}$ smooth. Thus we proved the assertion (1). If $\alpha_{n-1} = 1$ and F' contains a (-2) component, the image of $E^{(n-1)}(1,1)$ would have non-negative self-intersection number after the contraction of C_{12} and the subsequently contractible components in F'. Hence the part F' contains no (-2) components and it consists only of one (-3) curve. If $\alpha_{n-1} = 2$, we have to make (the image of) $E^{(n-1)}(1,1)$ a (-1) curve by the contraction of C_{12} and the (-2) components in the part F' following after the contractions of the part $C_{11}+B_1$ and the components $B', E^{(n-1)}(1), \cdots, E^{(n-1)}(r_{n-1} - 1)$ successively in this order (if any). It then follows that the number of (-2) curves contained in the part F' is $q_2^{(n-1)} - r_{n-1} - 1$. Q.E.D.

We can construct the surface V and the $\mathbf{P}^1$-fibration with the specific singular fibers $\widetilde{F_1}$ and $\widetilde{F_2}$ as follows. Let the notations $\pi : \Sigma_1 \to \mathbf{P}^1, l_1, l_2, M_1, M_2, Q_1$ and Q_2 be the same as those given after the proof of Lemma 4.3, where M_1 is the minimal section and $Q_i = l_i \cap M_2$ for $i = 1, 2$. Let θ_0 be the blowing-ups with centers at Q_1 and Q_2, and let $Q'_i := l'_i \cap \theta_0^{-1}(Q_i)$, where $l'_i := \theta'_0(l_i)$ for $i = 1, 2$.

We consider first the case C_{12} meets a component in the part B_3, i.e., C_{12} meets either $E(2, q_2)$ or $E^{(n-1)}(r_{n-1})$ (cf. Lemma 4.9).

(1) Suppose C_{12} meets the component $E(2, q_2)$. In order to produce the fiber $\widetilde{F_1}$, we perform the oscillating transformation θ_1 associated with

$(Q'_1, G'; q_\alpha, \cdots, q_4, q_3 - 1)$ (cf. Section 1), where $G' = \theta_0^{-1}(Q_1)$ if α is even and $G' = l'_1$ if α is odd, and let $\theta = \theta_0 \cdot \theta_1$. Let R' be the component with self-intersection number $-(1 + q_3)$ in the fiber $\theta^*(l_1)$. With the notations in the proof of Lemma 4.9, the configuration of $\theta^*(l_1)$ corresponds to the one of the linear chain $L + C_{12} + E'$, and we can make the connected part B which sprouts out of $L + C_{12} + E'$ by a succession of blowing-ups starting with the blowing-up with center at a point on R' and not lying on other components. Let ξ be this process.

(2) Suppose C_{12} meets the component $E^{(n-1)}(r_{n-1})$. In order to produce the fiber $\widetilde{F_1}$, we perform the oscillating transformation θ_1 associated with $(Q'_1, G'; q_\alpha, \cdots, q_2, q_1 - 2)$, where G' is the same as in the above case (1). Let $\theta = \theta_0 \cdot \theta_1$. Let R' be the component with self-intersection number $-q_1$ in the fiber $\theta^*(l_1)$. Let ξ be the same process as above to produce the connected part B.

(3) To produce the fiber $\widetilde{F_2}$, we perform the oscillating transformation associated with $(Q'_2, H'; q_\alpha, \cdots, q_2, q_1 - 1)$, where $H' = l'_2$ if α is even and $H' = \theta_0^{-1}(Q_2)$ if α is odd. By the abuse of notations, we assume hereon that θ includes this oscillating transformation to produce $\widetilde{F_2}$.

Let $\varrho = \theta \cdot \xi$. Then the fiber $\varrho^*(l_i)$ has the same configuration as the fiber $\widetilde{F_i}$ for $i = 1, 2$, and the images of the unique (-1) components in the fibers $\theta^*(l_1)$ and $\theta^*(l_2)$ are respectively the components C_{12} and C_2. Furthermore, the image of R' is the component H in A' meeting C_{12}.

By the above construction and Lemma 1.5, we can show that the multiplicities of the components C_{12} and C_2 in the fibers $\widetilde{F_1}$ and $\widetilde{F_2}$ are equal to $d_0 - d_1$ and d_0, respectively.

We consider next the case C_{12} meets the component $E^{(n-1)}(1,1)$. Since the construction of the fiber $\widetilde{F_2}$ from l_2 is the same as in the case C_{12} meets either $E(2, q_2)$ or $E^{(n-1)}(r_{n-1})$, we consider below only the construction of the fiber $\widetilde{F_1}$ from l_1. To simplify the notations, we put $r := r_{n-1}$ and $q := q_2^{(n-1)}$ (resp. $q := 1$) if $\alpha_{n-1} = 2$ (resp. $\alpha_{n-1} = 1$).

In order to produce the fiber $\widetilde{F_1}$, we perform the oscillating transformation θ_1 associated with $(Q'_1, G'; q_\alpha, \cdots, q_2, q_1 - 1, 1, q - (r+1))$, where $G' = \theta_0^{-1}(Q_1)$ if α is even and $G' = l'_1$ if α is odd. Note that $q - (r+1) \geq 0$ by Lemma 4.10, (1). Let R' and S' be the components with self-intersection number

$-(1+q_1)$ and $-(1+q)+r$ in the fiber $(\theta_0 \cdot \theta_1)^*(l_1)$, respectively. Let T' be the last exceptional component in the process $\theta_0 \cdot \theta_1$. We put $Q_1'' := R' \cap S'$ and perform the oscillating transformation θ_2 associated with $(Q_1'', S'; r)$. Set $\theta := \theta_0 \cdot \theta_1 \cdot \theta_2$ and let U' be the last component in the process θ. With the notations as in the proof of Lemma 4.10, the configuration of $\theta^*(l_1)$ corresponds to the one of the linear chain $L + C_{12} + E'$, and one can make the connected part $C_{11} + B_1$ which sprouts out of $L + C_{12} + E'$ by a succession of blowing-ups starting with the blowing-up with center at a point on U' and not lying on other components. Let ξ be this process and set $\varrho := \theta \cdot \xi$. Then the fiber $\varrho^*(l_1)$ has the same configuration as the fiber $\widetilde{F_1}$ and an image of T' is C_{12}. By the above construction and Lemma 1.5, we can show that the multiplicity of the component C_{12} in the fiber $\widetilde{F_1}$ is equal to $(q-r+1)d_0 - d_1$.

Now we shall determine the defining polynomial of the curve C by finding the polynomials in $\mathbf{C}[y,z]$, say f_{12} and f_2, to define F_{12}° and F_2° on $\mathbf{A}^2 = \mathbf{P}^2 - \overline{F_{11}} = \operatorname{Spec} \mathbf{C}[y,z]$.

We first consider the case where C_{12} meets either $E(2, q_2)$ or $E^{(n-1)}(r_{n-1})$. We contract all the components in $\sigma^{-1}(p_1, p_2) \cup C_{11} - E^{(n-1)}(r_{n-1})$ by starting with the contractions of H_1, H_2 and C_{11}. Let $\rho : V \to \mathbf{P}^2$ be this contraction. Let $l'_\infty = \rho(E^{(n-1)}(r_{n-1}))$, which is a line. Then a composite

$$\zeta : \mathbf{P}^2 \xrightarrow{\sigma^{-1}} V \xrightarrow{\rho} \mathbf{P}^2$$

is a Cremona transformation which induces the identity morphism between $\mathbf{P}^2 - \overline{F_{11}}$ and $\mathbf{P}^2 - l'_\infty$. Let (X', Y', Z') be a system of homogeneous coordinates on $\mathbf{P}^2$ such that l'_∞ is defined by $X' = 0$. Let $l'_{12} = \rho(C_{12}) = \zeta(\overline{F_{12}})$ and $l'_2 = \rho(C_2) = \zeta(\overline{F_2})$. Then we prove the following result.

Lemma 4.11 *Suppose that C_{12} meets either $E(2, q_2)$ or $E^{(n-1)}(r_{n-1})$. After a suitable choice of (X', Y', Z'), we may write the polynomial f_2 as $f_2 = z'$ and the polynomial f_{12} as*

$$f_{12} = \begin{cases} cy' + g(z') & \text{if } C_{12} \text{ meets } E(2, q_2) \\ y' & \text{if } C_{12} \text{ meets } E^{(n-1)}(r_{n-1}), \end{cases}$$

where $y' = Y'/X'$, $z' = Z'/X'$, $c \in \mathbf{C}^*$ and $\deg_{z'} g(z') = q_2 + 1$.

Proof. When we contract all the exceptional components of

$$(\sigma_1^{(1)} \cdot \tau_1^{(1)} \cdots \sigma_1^{(n-1)} \cdot \tau^{(n-1)})^{-1}(p_1) \bigcup C_{11} - E^{(n-1)}(r_{n-1}),$$

the image of the fiber $\widetilde{F_1}$ has the same configuration as the one in Figure 12 (resp. Figure 6) if C_{12} meets $E(2, q_2)$ (resp. $E^{(n-1)}(r_{n-1})$), where the image of $E^{(n-1)}(r_{n-1})$ replaces C_{11}. Successively, we contract all the exceptional components from the last Euclidean transformation $\sigma_1^{(n)}$ and the components from $\sigma^{-1}(p_2)$. Since C_{12} and C_2 meet the end components of the linear chain Supp $(\sigma^{-1}(p_2))$ (see Lemma 4.2), their images l'_{12} and l'_2 intersect each other transversally in a point of $\mathbf{P}^2 - l'_\infty = \operatorname{Spec} \mathbf{C}[y', z']$. Suppose that C_{12} meets the component $E(2, q_2)$. By Lemma 4.9, (1), we then have $q_1 = 1$ and $\alpha > 2$. When the component $E(3, 1)$ is contracted in the course of contracting the exceptional components of $\sigma_1^{(n)}$, we have the dual graph in Figure 14, where the components from the left to the right are respectively the images of $E(1,1), E(2, q_2), E(2, q_2 - 1), \cdots, E(2,1), E^{(n-1)}(r_{n-1})$.

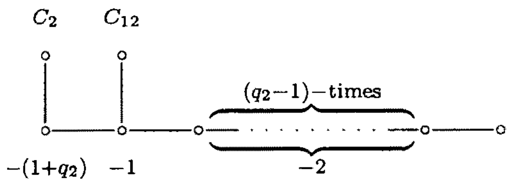


Figure 14:

It then follows that l'_{12} and l'_2 intersect the line l'_∞ with respective order $q_2 + 1$ and 1. Hence we may assume that the polynomial f_2 is written as $f_2 = z'$ and, moreover, we may assume that the polynomial f_{12} is written as

$$f_{12} = cy' + g(z'),$$

where $c \in \mathbf{C}^*, \deg g(z') = q_2 + 1$ (see the same argumemt before Theorem 4.5). Suppose that C_{12} meets the component $E^{(n-1)}(r_{n-1})$. Then $q_1 > 1$ by Lemma 4.9, (1). When we contract the exceptional components of $\sigma_1^{(n)}$, we know that the curves l'_{12} and l'_2 intersect transversally the line l'_∞ at distinct points. Hence we may assume that l'_{12} and l'_2 are defined by $Y' = 0$ and $Z' = 0$, respectively. So, we may assume that $f_{12} = y'$ and $f_2 = z'$. Q.E.D.

Let $L_{Y'}$ and $L_{Z'}$ be the lines defined by $Y' = 0$ and $Z' = 0$, respectively. We consider the inverse $\eta := \zeta^{-1}$ of the Cremona transformation $\zeta : \mathbf{P}^2 \to \mathbf{P}^2$, which induces a biregular automorphism $\eta : \operatorname{Spec}\mathbf{C}[y', z'] \to \operatorname{Spec}\mathbf{C}[y, z]$. We consider how y', z' are expressed as polynomials in y, z.

Lemma 4.12 *Assume that $n \geq 2$. For $0 \leq j < n$, define polynomials y_j and z_j in $\mathbf{C}[y, z]$ inductively as follows:*

$$y_0 := y \quad z_0 := z,$$

and

$$\begin{cases} y_j := y_{j-1} + g_{j-1}(z_{j-1}) \\ z_j := y_{j-1} + c_j z_{j-1} + g_{j-1}(z_{j-1}) \end{cases} \quad \text{for } 1 \leq j < n,$$

where $c_j \in \mathbf{C}^*, \deg_{z_{j-1}} g_{j-1}(z_{j-1}) = r_j + 1$ and $g_{j-1}(0) = (dg_{j-1}/dz_{j-1})(0) = 0$. Then we may assume that $y' = y_{n-1}$ and $z' = z_{n-1}$.

Proof. We prove the assertion by induction on n. Suppose $n = 2$. With the notations preceding this lemma, the curves $\eta(L_{Y'})$ and $\eta(L_{Z'})$ have the point p_1 on $\overline{F_{11}}$ in common, where both curves meet $\overline{F_{11}}$ with the same order $r_1 + 1$ and the same multiplicity r_1. Meanwhile, they intersect each other in the point p_2 on $\mathbf{A}^2 = \mathbf{P}^2 - \overline{F_{11}}$ transversally. We choose homogeneous coordinates (X, Y, Z) such that $p_1 = (0 : 1 : 0)$ and $p_2 = (1 : 0 : 0)$ and that the curve $\eta(L_{Z'})$ intersects the Y-axis at p_2 transversally. From these conventions concerning the coordinates in neighborhoods of p_1 and p_2, it follows that the polynomials y' and z' are respectively written as:

$$\begin{aligned} y' &= y + c_1 z + \ldots + c_{r_1} z^{r_1} + c_{r_1+1} z^{r_1+1} \\ z' &= y + c'_1 z + \ldots + c'_{r_1} z^{r_1} + c'_{r_1+1} z^{r_1+1}, \end{aligned}$$

where c_{r_1+1}, c'_{r_1+1} and c'_1 are non-zero. Note that the jacobian determinant $J((y', z')/(y, z))$ is a non-zero constant because of $\mathbf{C}[y', z'] = \mathbf{C}[y, z]$. Hence we have $c_1 \neq c'_1$ and $c_j = c'_j$ for $2 \leq j \leq r_1 + 1$. Hence, after replacing y by $y + c_1 z$ if $c_1 \neq 0$, we may assume that y' and z' are written as in the stated form.

Suppose now $n > 2$. We contract the components of

$$\sigma^{-1}(p_1) \bigcup C_{11} - E^{(1)}(r_1),$$

starting with the contractions of C_{11} and H_1. Successively we contract the part $\operatorname{Supp} \sigma^{-1}(p_2)$ and denote by $\rho' : V \to \mathbf{P}^2$ a composite of the above contractions. Let $\overline{l_\infty} = \rho'(E^{(1)}(r_1))$, which is a line. Then we obtain a Cremona transformation

$$\eta' : \mathbf{P}^2 \xrightarrow{\rho^{-1}} V \xrightarrow{\rho'} \mathbf{P}^2,$$

which induces a biregular automorphism η' : Spec $\mathbf{C}[y', z'] \to$ Spec $\mathbf{C}[\overline{y}, \overline{z}]$, where we choose a system of homogeneous coordinates $(\overline{X}, \overline{Y}, \overline{Z})$ on the right $\mathbf{P}^2$ such that the line $\overline{l_\infty}$ is defined by $\overline{X} = 0$ and where $\overline{y} = \overline{Y}/\overline{X}, \overline{z} = \overline{Z}/\overline{X}$. By the inductive hypothesis, we may write $y' = \overline{y_{n-2}}$ and $z' = \overline{z_{n-2}}$, where polynomials $\overline{y_j}, \overline{z_j}$ of $\mathbf{C}[\overline{y}, \overline{z}]$ for $0 \le j < n-1$ are defined as follows:

$$\overline{y_0} := \overline{y} \quad \overline{z_0} := \overline{z},$$

$$\begin{cases} \overline{y_j} & := & \overline{y_{j-1}} + \overline{g_{j-1}}(\overline{z_{j-1}}) \\ \overline{z_j} & := & \overline{y_{j-1}} + c_{j-1}\overline{z_{j-1}} + \overline{g_{j-1}}(\overline{z_{j-1}}) \end{cases} \quad \text{for } 1 \le j < n-1,$$

where $c_{j-1} \in \mathbf{C}^*, \deg \overline{g_{j-1}}(\overline{z_{j-1}}) = r_{j+1} + 1$ and $\overline{g_{j-1}}(0) = (d\overline{g_{j-1}}/d\overline{z_{j-1}})(0) = 0$. We now reproduce the part $(\sigma_1^{(1)} \cdot \tau_1^{(1)})^{-1}(p_1) \cup C_{11}$ by a succession of blowing-ups which starts with the blowing-up with center on $\overline{l_\infty}$ and successively contract all the components of it except for C_{11}. Then we obtain a Cremona transformation η'' satisfying $\eta = \eta'' \cdot \eta'$, which induces a biregular automorphism η'' : Spec $\mathbf{C}[\overline{y}, \overline{z}] \to$ Spec $\mathbf{C}[y, z]$. By the same argumemt as in the case $n = 2$, we may write $\overline{y} = y_1$ and $\overline{z} = z_1$, respectively. Therefore, we may assume that y' and z' are written as y_{n-1} and z_{n-1}, respectively. Q.E.D.

As a consequence of Lemmas 4.11, 4.12, we have the following theorem:

Theorem 4.13 *Suppose that σ_1 is written as $\sigma_1 = \sigma_1^{(1)} \cdot \tau_1^{(1)} \cdots \sigma_1^{(n-1)} \cdot \tau_1^{(n-1)} \cdot \sigma_1^{(n)}$ with $n \ge 2$ and , furthermore, that the component C_{12} meets $E(2, q_2)$ or $E^{(n-1)}(r_{n-1})$. Then $C^\circ = C - \{p_1\}$ is defined by a polynomial f on* $\mathbf{P}^2 - \overline{F_{11}} =$ Spec $\mathbf{C}[y, z]$ *of the following form:*

$$f = \begin{cases} (cy' + g(z'))^{d_2} + \lambda z'^{d_0} & \text{if } C_{12} \text{ meets } E(2, q_2) \\ y'^{d_0 - d_1} + \lambda z'^{d_0} & \text{if } C_{12} \text{ meets } E^{(n-1)}(r_{n-1}), \end{cases}$$

where the polynomials y' and z' are those given in Lemma 4.12 and where $\lambda, c \in \mathbf{C}^*, \deg g(z') = q_2 + 1$.

REMARK **4.14** Though we proved Theorem 4.13 under the assumption $n \geq 2$ it is clear that the theorem is valid also in the case where $n = 1$ (cf. Theorem 4.5). Note that if $n = 1$, the component C_{12} meets $E(2, q_2)$ (cf. Figure 12).

We consider finally the case where C_{12} meets $E^{(n-1)}(1,1)$ and determine the defining polynomial of the curve C. We contract all the components in $\sigma^{-1}(p_1, p_2) \cup C_{11} - E^{(n-2)}(r_{n-2})$ (we put $E^{(0)}(r_0) := C_{11}$ for $n = 2$) starting with the contractions of H_1, H_2 and C_{11}. Let $\varepsilon : V \longrightarrow \mathbf{P}^2$ be this contraction and let $l''_\infty := \varepsilon(E^{(n-2)}(r_{n-2}))$, which is a line. Then a composite

$$\vartheta : \mathbf{P}^2 \xrightarrow{\sigma^{-1}} V \xrightarrow{\varepsilon} \mathbf{P}^2$$

is a Cremona transformation which induces a biregular automorphism between $\mathbf{P}^2 - \overline{F_{11}}$ and $\mathbf{P}^2 - l''_\infty$. Let (X'', Y'', Z'') be a system of homogeneous coordinates on the right $\mathbf{P}^2$ such that the line l''_∞ is defined by $X'' = 0$. Let $l''_{12} := \varepsilon(C_{12}) = \vartheta(\overline{F_{12}})$ and $l''_2 := \varepsilon(C_2) = \vartheta(\overline{F_2})$. Then we prove the following result analogous to Lemma 4.11:

Lemma 4.15 *Suppose that C_{12} meets $E^{(n-1)}(1,1)$. After a suitable choice of (X'', Y'', Z''), we may write the polynomials f_{12} and f_2 as $f_{12} = z''$ and $f_2 = cy'' + g(z'')$, where $y'' := Y''/X''$, $z'' := Z''/X''$, $c \in \mathbf{C}^*$ and $\deg_{z''} g(z'') = q + 1$.*

Proof. In the course of the process ε, we contract all the components in

$$C_{11} \bigcup \left(\sigma_1^{(1)} \cdot \tau_1^{(1)} \cdots \sigma_1^{(n-2)} \cdot \tau_1^{(n-2)}\right)^{-1}(p_1) \bigcup \left(\tau_1^{(n-1)} \cdot \sigma_1^{(n)}\right)^{-1}(Q) - E^{(n-2)}(r_{n-2}),$$

where $\tau_1^{(n-1)}$ starts with the blowing-up with center Q. We have the dual graph in Figure 15, where the components from the left to the right are respectively the images of $E^{(n-1)}(1,1), E^{(n-1)}(2,q), \cdots, E^{(n-1)}(2,1), E^{(n-2)}(r_{n-2})$ if $\alpha_{n-1} = 2$. They are the images of $E^{(n-1)}(1,1), E^{(n-1)}(1,2), E^{(n-2)}(r_{n-2})$ if $\alpha_{n-1} = 1$.

It then follows that l''_{12} and l''_2 intersect the line l''_∞ with respective order of contact 1 and $q + 1$. Hence we may write the polynomials f_{12} and f_2 as

$$f_{12} = z'' \quad \text{and} \quad f_2 = cy'' + g(z''),$$

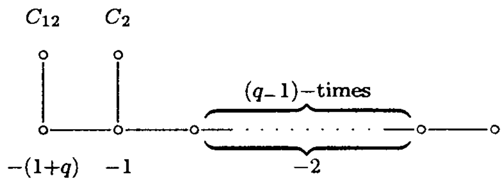


Figure 15:

where $c \in \mathbf{C}^*$ and $\deg_{z''} g(z'') = q + 1$ (see the argument before Theorem 4.5). Q.E.D.

The argument in the proof of Lemma 4.12 implies that we may write y'' and z'' as polynomials in the affine coordinates (y, z). More precisely, $y'' = y_{n-2}$ and $z'' = z_{n-2}$ as in Lemma 4.12. Summarizing these observations and Lemma 4.15, we have the following result.

Theorem 4.16 *Suppose that $\sigma_1 = \sigma_1^{(1)} \cdot \tau_1^{(1)} \cdots \sigma_1^{(n-1)} \cdot \tau_1^{(n-1)} \cdot \sigma_1^{(n)}$ with $n \geq 2$ and that the component C_{12} meets $E^{(n-1)}(1,1)$. Then the curve $C^\circ = C - \{p_1\}$ on $\mathbf{P}^2 - \overline{F_{11}} = \operatorname{Spec} \mathbf{C}[y,z]$ is defined by a polynomial f of the following form:*

$$f = z''^{(q-r+1)d_0 - d_1} + \lambda(cy'' + g(z''))^{d_0},$$

where y'', z'' are polynomials in $\mathbf{C}[y,z]$ as specified as above, $\lambda, c \in \mathbf{C}^*$, $\deg_{z''} g(z'') = q+1$, $r := r_{n-1}$ and $q := q_2^{(n-1)}$ (resp. $q := 1$) if $\alpha_{n-1} = 2$ (resp. $\alpha_{n-1} = 1$).

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